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ROY DUBISCH, *Editor*

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GENERALIZATIONS OF THEOREMS ABOUT TRIANGLES

CARL B. ALLENDOERFER, University of Washington

1. Introduction. Since one of the most powerful methods in mathematical research is the process of generalization, it is certainly desirable that young students be introduced to this process as early as possible. The purpose of this article is to call attention to the usually untapped possibilities for generalizing theorems on the triangle to theorems about the tetrahedron. Some of these, of course, do appear in our textbooks on solid geometry; but here I shall describe two situations where the appropriate generalizations seem to be generally unknown. The questions to be answered are: (1) What is the generalization to a tetrahedron of the angle-sum theorem for a triangle? (2) What is the corresponding generalization of the laws of sines and cosines for a triangle? Expressed in this form, the questions are certainly vague; for surely there are many generalizations. From these we are to select the ones which are most satisfying and which have a clear right to be called *the generalizations*. In attacking these problems we will need to reexamine the theorems as they are stated for a triangle, and perhaps to reformulate them so that the generalizations appear to be natural. Thus we have a bonus in that we learn additional ways of thinking about triangles.

2. The angle-sum theorem. Since this theorem is one of the most familiar in Euclidean geometry, it is strange that its three-dimensional generalization is not part of the classical literature on geometry. I ran across this generalization some years ago and have been putting the question to mathematicians wherever I find them. Only one of them, Professor Pólya, knew of it. He attributes it to Descartes [1].

The first question to be settled is that of the type of angles in a tetrahedron to be considered. It would be most natural to consider the inner solid angles and their sum. I remind you that the measure of a solid angle is the area of the region on the unit sphere which is the intersection of the sphere with the interior of the solid angle whose vertex is at the center. Thus the measure of the solid angle at a corner of a room is $4\pi/8=\pi/2$, and the measure of a "straight" solid angle is $4\pi/2=2\pi$. By considering a few cases, we conclude that the sum of the measures of the inner solid angles of a tetrahedron is not a constant. For example consider the situation in Figure 1, where all the points lie in a plane. If D is raised slightly, we have a tetrahedron the sum of whose interior solid angles is very near to 2π. On the other hand let us raise segment AB in the plane Figure 2 a small amount. Then we have a tetrahedron the sum of whose inner solid angles is very near to zero. Hence the obvious generalization is incorrect. As a matter of fact it has been proved [2] that the sum of the solid angles of a tetrahedron can take any value between 0 and 2π.

In order to make a fresh start, let us reformulate the triangle theorem in the statement: The sum of the outer angles of a triangle equals 2π. There are two possible definitions of an outer angle. The usual one is that it is the angle between a pair of successive directed sides (Fig. 3). This clearly does not general-

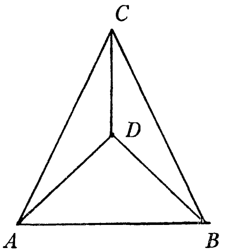


FIG. 1.

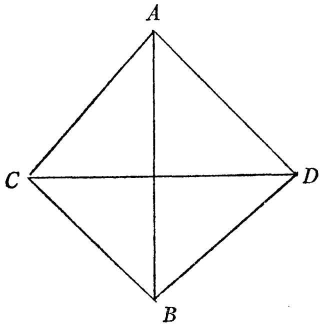
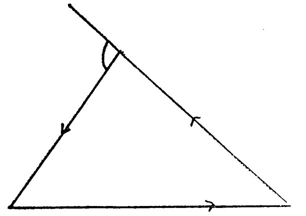


FIG. 2.



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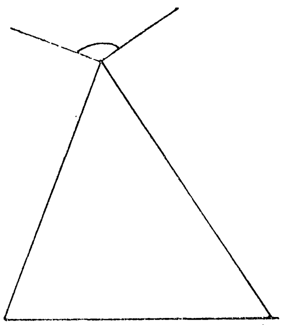

FIG. 4.

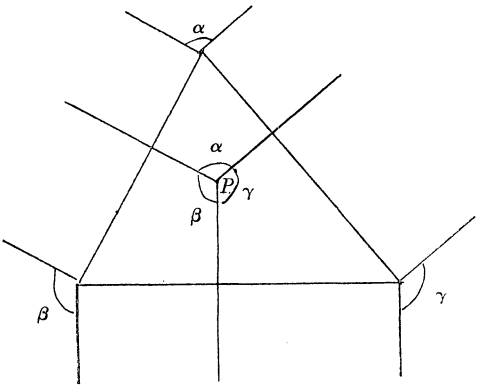

FIG. 5.

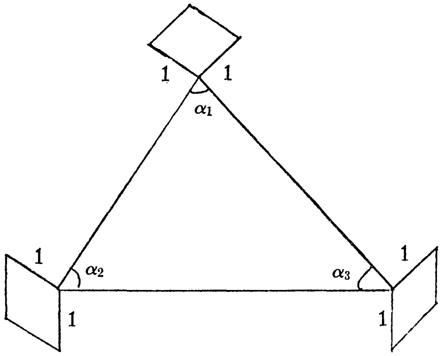

FIG. 6.

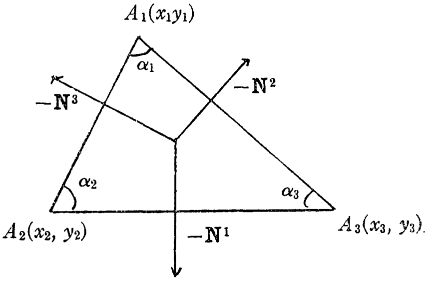

FIG. 7.

ize to three dimensions. Less familiar is the definition that an outer angle at a vertex is the angle between the two outward drawn normals to the two edges which meet at this vertex (Fig. 4).

Using this second definition, we can construct an elegant proof of the theorem. Choose any point P in the interior of the triangle and draw the perpendiculars from P to the three sides (Fig. 5). Then the outer angles α, β, and γ are equal to the three angles formed at P. Hence $\alpha+\beta+\gamma=2\pi$.

Now we can generalize at once. To find the corresponding theorem on the tetrahedron, first define the outer angle at a vertex as the trihedral angle formed by the three outer normals to the three faces meeting at this vertex. Choose an interior point P and draw the perpendiculars from P to the four faces. By the same argument that we used for the triangle, we find that

THEOREM 1. *The sum of the outer angles of a tetrahedron is* 4π.

By a straightforward generalization of the notion of an outer angle, we can similarly prove that

THEOREM 2. *The sum of the outer angles of any convex polyhedron is equal to* 4π.

There is also an immediate generalization to higher dimensions.

3. The Laws of Sines and Cosines. Before considering the generalization of these laws to a tetrahedron, let me give unfamiliar proofs of them which will suggest the proper generalization.

First, consider the Law of Sines. At each vertex (Fig. 6) draw the unit outer normals to the sides meeting at that vertex and complete the parallelograms determined by these pairs. By a familiar theorem of trigonometry the areas of these parallelograms are respectively $\sin(\pi-\alpha_1)=\sin\alpha_1$, $\sin(\pi-\alpha_2)=\sin\alpha_2$, and $\sin(\pi-\alpha_3)=\sin\alpha_3$. We shall proceed to compute these areas in terms of the coordinates of the vertices of the triangle (Fig. 7), choosing the notation appropriately so that $A_1A_2A_3$ are labeled in a counterclockwise fashion.

The equation of side A_2A_3 is

$$xN_x^1 + yN_y^1 + (x_2y_3 - x_3y_2) = 0$$

where N_x^1 and N_y^1 are respectively the cofactors of x_1 and y_1 in the determinant

$$\Delta = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

Thus the vector $\mathbf{N}^1$ with components (N_x^1, N_y^1) is normal to A_2A_3; $-\mathbf{N}^1$ is an outer normal; and $\mathbf{U}^1=-\mathbf{N}^1/a_1$ is the unit outer normal (where a_1 is the length of A_2A_3). More generally $\mathbf{U}^i=-\mathbf{N}^i/a_i$ $(i=1, 2, 3)$ are the three outer normals, where N_x^i and N_y^i are the cofactors of x_i and y_i respectively and a_i is the length of the side to which $\mathbf{U}^i$ is normal.

The area of the outer parallelogram at A_1 of which two sides are $\mathbf{U}^2$ and $\mathbf{U}^3$ is

$$\sin \alpha_1 = \begin{vmatrix} U_x^2 & U_y^2 \\ U_x^3 & U_y^3 \end{vmatrix} = \frac{1}{a_2 a_3} \begin{vmatrix} N_x^2 & N_y^2 \\ N_x^3 & N_y^3 \end{vmatrix}.$$

By a classical theorem on determinants (Bôcher, Introduction to Higher Algebra, p. 31) it follows that

$$\begin{vmatrix} N_x^2 & N_y^2 \\ N_x^3 & N_y^3 \end{vmatrix} = \Delta \cdot 1.$$

Hence

$$\sin \alpha_1 = \frac{\Delta}{a_2 a_3} \quad \text{and} \quad \frac{\sin \alpha_1}{a_1} = \frac{\Delta}{a_1 a_2 a_3}.$$

In a similar fashion we prove that

$$\frac{\sin \alpha_1}{a_1} = \frac{\sin \alpha_2}{a_2} = \frac{\sin \alpha_3}{a_3} = \frac{\Delta}{a_1 a_2 a_3}$$

which is the familiar Law of Sines.

To arrive at the Law of Cosines, we begin with a theorem of Möbius.

THEOREM 3. $\mathbf{N}^1 + \mathbf{N}^2 + \mathbf{N}^3 = \mathbf{0}$.

This theorem follows from the facts that $N_x^1 + N_x^2 + N_x^3 = 0$ and $N_y^1 + N_y^2 + N_y^3 = 0$. These may be computed directly, or they may be proved by expanding the determinants

$$\begin{vmatrix} 1 & y_1 & 1 \\ 1 & y_2 & 1 \\ 1 & y_3 & 1 \end{vmatrix} = 0 \quad \text{and} \quad \begin{vmatrix} x_1 & 1 & 1 \\ x_2 & 1 & 1 \\ x_3 & 1 & 1 \end{vmatrix} = 0.$$

This theorem can be rewritten in the form:

$$\mathbf{N}^1 = -\mathbf{N}^2 - \mathbf{N}^3.$$

Now take the scalar product of each side of this equation with itself. The result is

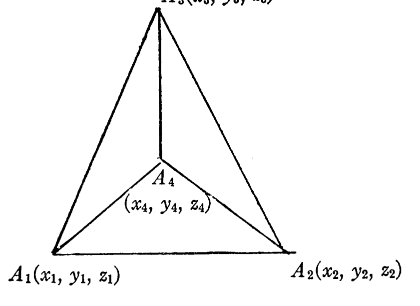

FIG. 8.

$$\mathbf{N}^1\cdot\mathbf{N}^1 = \mathbf{N}^2\cdot\mathbf{N}^2 + \mathbf{N}^3\cdot\mathbf{N}^3 + 2\mathbf{N}^2\cdot\mathbf{N}^3.$$

Since $\mathbf{N}^i\cdot\mathbf{N}^i=a_i^2$, and $\mathbf{N}^2\cdot\mathbf{N}^3=-a_2a_3\cos\alpha_1$, this becomes $a_1^2=a_2^2+a_3^2-2a_2a_3\cos\alpha_1$.

4. The Generalized Laws of Sines and Cosines. These generalizations are due to Grassmann, but are relatively unfamiliar. Their proofs follow the lines just given in Section 3.

Consider a tetrahedron (Fig. 8) whose vertices are ordered so that

$$\Delta = \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} > 0.$$

Then the vector $\mathbf{N}^1$ whose components (N_x^1, N_y^1, N_z^1), are the cofactors of x_1, y_1, z_1 respectively in Δ, is normal to the face $A_2A_3A_4$. The length of $\mathbf{N}^1$, namely a_1, is equal to twice the area of this face. The vector $\mathbf{U}^1=-\mathbf{N}^1/a_1$ is the unit outer normal to this face. Other normals $\mathbf{N}^i$ and $\mathbf{U}^i$ are defined in a similar fashion.

We now define the generalized sine ("G-sin") of the inner trihedral angle at A_1 to be the volume of the parallelopiped whose edges are $\mathbf{U}^2$, $\mathbf{U}^3$, and $\mathbf{U}^4$. Thus

$$G\text{-}\sin\alpha_1 = \begin{vmatrix} U_x^2 & U_y^2 & U_z^2 \\ U_x^3 & U_y^3 & U_z^3 \\ U_x^4 & U_y^4 & U_z^4 \end{vmatrix} = \frac{-1}{a_2a_3a_4}\begin{vmatrix} N_x^2 & N_y^2 & N_z^2 \\ N_x^3 & N_y^3 & N_z^3 \\ N_x^4 & N_y^4 & N_z^4 \end{vmatrix} = \frac{(-1)\Delta^2(-1)}{a_2a_3a_4} = \frac{\Delta^2}{a_2a_3a_4}.$$

By a continuation of this argument, we obtain the Generalized Law of Sines:

THEOREM 4.

$$\frac{G\text{-}\sin\alpha_1}{a_1} = \frac{G\text{-}\sin\alpha_2}{a_2} = \frac{G\text{-}\sin\alpha_3}{a_3} = \frac{G\text{-}\sin\alpha_4}{a_4} = \frac{\Delta^2}{a_1a_2a_3a_4}.$$

To establish the Generalized Law of Cosines, we observe that we can prove the following generalization of the Theorem of Möbius.

THEOREM 5. $\mathbf{N}^1+\mathbf{N}^2+\mathbf{N}^3+\mathbf{N}^4=0$.

Then writing

$$\mathbf{N}^1 = -\mathbf{N}^2 - \mathbf{N}^3 - \mathbf{N}^4$$

and $f_i=a_i/2=$ area of the ith face, we prove as above the result:

THEOREM 6.

$$f_1^2 = f_2^2 + f_3^2 + f_4^2 - 2[f_2f_3\cos(f_2,f_3) + f_2f_4\cos(f_2,f_4) + f_3f_4\cos(f_3,f_4)],$$

where (f_i, f_j) is the inner dihedral angle of the tetrahedron between the faces whose areas are f_i and f_j respectively.

We also have another, rather novel, generalization if we start from $\mathbf{N}^1+\mathbf{N}^2=-\mathbf{N}^3-\mathbf{N}^4$. The result is

THEOREM 7.

$$f_1^2+f_2^2-2f_1f_2\cos(f_1,f_2)=f_3^2+f_4^2-2f_3f_4\cos(f_3,f_4).$$

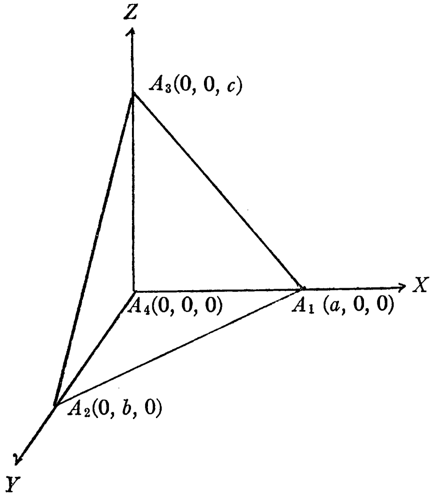

FIG. 9.

5. Supplementary matters. Another approach to the Generalized Law of Sines is to begin with a right tetrahedron (Fig. 9). Then it would be reasonable to define

$$G\text{-}\sin\alpha_1=\frac{\text{Area } A_2A_3A_4}{\text{Area } A_1A_2A_3}=\frac{bc}{\{b^2c^2+a^2c^2+a^2b^2\}^{1/2}}.$$

Let us show that this agrees with our previous definition of G-sin α_1. We have:

$$\Delta=\begin{vmatrix} a & 0 & 0 & 1\\ 0 & b & 0 & 1\\ 0 & 0 & c & 1\\ 0 & 0 & 0 & 1\end{vmatrix}.$$

Then

$$G\text{-}\sin\alpha_1=\frac{\Delta^2}{a_2a_3a_4}=\frac{a^2b^2c^2}{(ac)(ab)\{b^2c^2+a^2c^2+a^2b^2\}^{1/2}}=\frac{bc}{\{b^2c^2+a^2c^2+a^2b^2\}^{1/2}}.$$

Also we have the reassuring result that for our right tetrahedron:

$$(G\text{-}\sin\alpha_1)^2+(G\text{-}\sin\alpha_2)^2+(G\text{-}\sin\alpha_3)^2=1.$$

It is natural to ask whether G-sin α_1 is actually the sine of the measure of the inner or the outer solid angle at A_1; the answer is "no". To give an elementary counter-example we consider the right tetrahedron with $a=b=c=1$. Then G-sin $\alpha_1=1/\sqrt{3}$; sine (measure of inner solid angle at A_1) $=1/3$; and sine (measure of outer solid angle at A_1) $=\sin 7\pi/6=-1/2$.

As a matter of fact, G-sin α is not even a functon of either the inner or the outer solid angles at the given vertex. Rather it depends directly on the face angles of the outer trihedral angle. If these angles are λ, μ, ν and $s=(\lambda+\mu+\nu)/2$, then

$$G\text{-}\sin \alpha = \{\sin s \sin (s-\lambda) \sin (s-\mu) \sin (s-\nu)\}^{1/2}.$$

References

1. R. Descartes, Oeuvres, vol. 10, pp. 257–276.

2. J. W. Gaddum, The sums of the dihedral and trihedral angles of a tetrahedron, Amer. Math. Monthly, 59 (1952) 370–375.

ON SUMS OF INVERSES OF PRIMES

J. H. JORDAN, Washington State University

1. Let P be the set of prime numbers and let $p_1=2$, $p_2=3$, $p_3=5$, $p_4=7$, etc. It is a well known and useful fact that the infinite series $\sum_{i=1} p_i^{-1}$ diverges. It is further known that the approximate rate at which $\sum_{i=1} p_i^{-1}$ diverges, is

$$\sum_{p_i \leqq x} p_i^{-1} = \ln \ln x + K + O(1/\ln x),$$

where K is a constant independent of x [1].

It seems natural to ask the following question: If $S \subseteq P$ does $\sum_{p_i \in S} p_i^{-1}$ converge or diverge?

The answer, of course, depends on S. For example, if $P-S$ is finite then surely $\sum_{p_i \in S} p_i^{-1}$ diverges; on the other hand, if S is finite $\sum_{p_i \in S} p_i^{-1}$ converges. The only case which has interest is when $P-S$ and S are both infinite.

For integers k and t, let $S(k, t)=\{kh+t\}_{h=1}^{\infty} \cap P$. Dirichlet proved that $S(k, t)$ is infinite if $(k, t)=1$, [1]. If $k>2$, then $P-S(k, t)$ is also infinite. It is known that $\sum_{p_i \in S(k,t)} p_i^{-1}$ diverges; in fact

$$\sum_{\substack{p_i \in S(k,t) \\ p_i \leqq x}} p_i^{-1} = (\phi(k))^{-1} \ln \ln x + K_1 + O(1/\ln x)$$

where K_1 is a constant and ϕ is the Euler phi function [2].

It may occur that S may be so defined that one is not able to say if S is finite or not. The convergence of $\sum_{p \in S} p^{-1}$ still allows that S could be finite but adds little credence to it being so. To illustrate this point consider Brun's Theorem; that is if S^* is the set of all twin primes then $\sum_{p \in S^*} p^{-1}$ converges [3].

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References

1. R. Descartes, Oeuvres, vol. 10, pp. 257–276.
2. J. W. Gaddum, The sums of the dihedral and trihedral angles of a tetrahedron, Amer. Math. Monthly, 59 (1952) 370–375.

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A further illustration is as follows: For $n \geqq 2$ let $F_n(x)$ be an nth degree irreducible polynomial with integer coefficients. If $S(F_n) = \{F_n(m)\}_{m=1}^{\infty} \cap P$ then $\sum_{p \in S(F_n)} p^{-1}$ converges. This can be established by a comparison test with $\sum_{n=1}^{\infty} n^{-2} = \pi^2/6$.

Other questions may occur such as: What can be said about the convergence of $\sum_{p \in S} p^{-1}$ if

(a) $S = \{p_{2m}\}_{m=1}^{\infty}$?
(b) $S = \{p_{am+b}\}_{m=1}^{\infty}$?
(c) $S = \{p_{p_m}\}_{m=1}^{\infty}$?
(d) $S = \{p_{a^m}\}_{m=1}^{\infty}$, a an integer?

It is the purpose of this paper to settle these four questions and other similar ones that may arise when S is defined in terms of the subscripts. That is if $S = \{p_{n_k}\}_{k=1}^{\infty}$, where n_k is a subsequence of the integers, necessary and sufficient conditions are placed on the n_k to assure the convergence of $\sum_{p \in S} p^{-1}$. The appropriate result is

THEOREM. *If* $S = \{p_{n_k}\}_{k=1}^{\infty}$ *then* $\sum_{p \in S} p^{-1}$ *converges if and only if* $\sum_{k=2}^{\infty} (n_k \ln n_k)^{-1}$ *converges.*

2. *Proof.* The following Lemma is proved in Hardy and Wright [4].

LEMMA. *There are positive constants* A_1 *and* A_2 *such that* $A_1 n \ln n < p_n < A_2 n \ln n$ *for* $n > N_0$.

Now if $\sum_{k=2}^{\infty} (n_k \ln n_k)^{-1}$ diverges then for $n_k > N_0$ or $k \geqq H$

$$\frac{p_{n_k}}{n_k \ln n_k} < \frac{A_2 n_k \ln n_k}{n_k \ln n_k} = A_2.$$

Hence $\sum_{k=H}^{\infty} (n_k \ln n_k)^{-1} \leqq A_2 \sum_{k=H}^{\infty} p_{n_k}^{-1}$; therefore $\sum_{k=2}^{\infty} p_{n_k}^{-1}$ diverges. If $\sum_{k=2}^{\infty} (n_k \ln n_k)^{-1}$ converges then, for $k > H$,

$$\frac{p_{n_k}}{n_k \ln n_k} \geqq \frac{A_1 n_k \ln n_k}{n_k \ln n_k} = A_1.$$

Therefore $\sum_{k=H}^{\infty} (n_k \ln n_k)^{-1} \geqq A_1 \sum_{k=H}^{\infty} p_{n_k}^{-1}$; hence $\sum_{k=1}^{\infty} p_{n_k}^{-1}$ converges.

Questions a, b, c, and d are now easily answered by the following corollaries.

COROLLARY 1. *If* $S = \{p_{2m}\}_{m=1}^{\infty}$ *then* $\sum_{p \in S} p^{-1}$ *diverges.*

Proof.

$$\sum_{m=1}^{\infty} (2m \ln 2m)^{-1} \geqq \sum_{m=2}^{\infty} (4m \ln m)^{-1} + (2 \ln 2)^{-1}$$

$$\geqq 1/4 \sum_{m=2}^{\infty} (m \ln m)^{-1} + (2 \ln 2)^{-1};$$

but $\sum_{m=2}^{\infty} (m \ln m)^{-1}$ is known to diverge by the integral test. Hence $\sum_{m=1}^{\infty} (2m \ln 2m)^{-1}$ diverges and therefore $\sum_{p \in S} p^{-1}$ diverges.

COROLLARY 2. *If* $S = \{p_{am+b}\}_{m=1}^{\infty}$, *then* $\sum_{p \in S} p^{-1}$ *diverges.*

Proof. Let $M=[b/a]+1$; then

$$\begin{aligned}\sum_{m=1}^{\infty} ((am+b)\ln(am+b))^{-1} \\ &\geqq \sum_{m=M}^{\infty} ((am+b)\ln(am+b))^{-1} + \sum_{m=1}^{M-1} ((am+b)\ln(am+b))^{-1} \\ &\geqq \sum_{m=M}^{\infty} (2am\ln 2am)^{-1} + \sum_{m=1}^{M-1} ((am+b)\ln(am+b))^{-1} \\ &\geqq \sum_{m=M}^{\infty} (4a^2 m\ln m)^{-1} + \sum_{m=1}^{M-1} ((am+b)\ln(am+b))^{-1} \\ &\geqq 1/4a^2 \sum_{m=M}^{\infty} (m\ln m)^{-1} + \sum_{m=1}^{M-1} ((am+b)\ln(am+b))^{-1}.\end{aligned}$$

But $1/4a^2\sum_{m=M}^{\infty} (m \ln m)^{-1}$ diverges; hence $\sum_{m=1}^{\infty} ((am+b)\ln(am+b))^{-1}$ diverges and so does $\sum_{p\in S} p^{-1}$.

COROLLARY 3. *If $S=\{p_{p_m}\}_{m=1}^{\infty}$, then $\sum_{p\in S} p^{-1}$ converges.*

Proof.

$$\begin{aligned}\sum_{m=1}^{\infty} (p_m \ln p_m)^{-1} &= \sum_{m=N_0+1}^{\infty} (p_m \ln p_m)^{-1} + \sum_{m=1}^{N_0} (p_m \ln p_m)^{-1} \\ &\leqq \sum_{m=N_0+1}^{\infty} (A_1 m\ln m(\ln(A_1 m\ln m)))^{-1} + \sum_{m=1}^{N_0} (p_m \ln p_m)^{-1} \\ &= \sum_{m=N_0+1}^{\infty} (A_1 m\ln m(\ln m + \ln A_1 \ln m))^{-1} + \sum_{m=1}^{N_0} (p_m \ln p_m)^{-1} \\ &\leqq \sum_{m=N_0+1}^{\infty} (A_1 m\ln^2 m)^{-1} + \sum_{m=1}^{N_0} (p_m \ln p_m)^{-1}.\end{aligned}$$

But $\sum_{m=N_0+1}^{\infty} (A_1 m \ln^2 m)^{-1}$ converges by the integral test so $\sum_{m=1}^{\infty} (p_m \ln p_m)^{-1}$ converges and so does $\sum_{p\in S} p^{-1}$.

COROLLARY 4. *If $S=\{p_{a^m}\}_{m=1}^{\infty}$, then $\sum_{p\in S} p^{-1}$ converges.*

Proof.

$$\sum_{m=1}^{\infty} (a^m \ln a^m)^{-1} \leqq \sum_{m=2}^{\infty} a^{-m} + (a\ln a)^{-1} \quad \text{and} \quad \sum_{m=1}^{\infty} a^{-m}$$

converges for $a\geqq 2$; hence $\sum_{p\in S} p^{-1}$ converges.

Two other corollaries which may be useful are:

COROLLARY 5. *If for $k>K_0$, $n_k \geqq k(\ln\ln k)^{1+\epsilon}$ for some $\epsilon>0$ and $S=\{p_{n_k}\}_{k=1}^{\infty}$, then $\sum_{p\in S} p^{-1}$ converges.*

Proof

$$\begin{aligned}\sum_{k=1}^{\infty}(n_k \ln n_k)^{-1} &= \sum_{k=K_0+1}^{\infty}(n_k \ln n_k)^{-1} + \sum_{k=1}^{K_0}(n_k \ln n_k)^{-1} \\ &\leqq \sum_{k=K_0+1}^{\infty}(k(\ln\ln k)^{1+\epsilon}(\ln(k(\ln\ln k)^{1+\epsilon})))^{-1} + \sum_{k=1}^{K_0}(n_k \ln n_k)^{-1} \\ &\leqq \sum_{k=K_0+1}^{\infty}(k(\ln\ln k)^{1+\epsilon}\ln k)^{-1} + \sum_{k=1}^{K_0}(n_k \ln n_k)^{-1}.\end{aligned}$$

But $\sum_{k=K_0+1}^{\infty}(k(\ln\ \ln\ k)^{1+\epsilon}\ \ln\ k)^{-1}$ converges by the integral test; hence $\sum_{k=1}^{\infty}(n_k\ \ln\ n_k)^{-1}$ converges and so does $\sum_{p\in S} p^{-1}$.

COROLLARY 6. *If for* $k>K_0$, $n_k \leqq k \ln \ln k$ *and* $S=\{p_{n_k}\}_{k=1}^{\infty}$, *then* $\sum_{p\in S} p^{-1}$ *diverges.*

Proof

$$\begin{aligned}\sum_{k=1}^{\infty}(n_k \ln n_k)^{-1} &\geqq \sum_{k=K_0+1}^{\infty}(n_k \ln n_k)^{-1} \\ &\geqq \sum_{k=K_0+1}^{\infty}(k \ln\ln k \ln(k\ln\ln k))^{-1} \\ &\geqq \sum_{k=K_0+1}^{\infty}(k\ln\ln k(2\ln k))^{-1},\end{aligned}$$

which diverges by the integral test.

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CONIC SECTIONS BY VECTOR METHODS

ARNOLD JOHNSON, University of Toledo

In the following we treat conic sections from a consistent vector viewpoint. This is rarely if ever done but since the advantages of a coordinate-free viewpoint are well known there seemed to be something to be gained and nothing to be lost by doing so.

Proof

$$\sum_{k=1}^{\infty}(n_k \ln n_k)^{-1} = \sum_{k=K_0+1}^{\infty}(n_k \ln n_k)^{-1} + \sum_{k=1}^{K_0}(n_k \ln n_k)^{-1}$$
$$\leqq \sum_{k=K_0+1}^{\infty}(k(\ln\ln k)^{1+\epsilon}(\ln(k(\ln\ln k)^{1+\epsilon})))^{-1} + \sum_{k=1}^{K_0}(n_k \ln n_k)^{-1}$$
$$\leqq \sum_{k=K_0+1}^{\infty}(k(\ln\ln k)^{1+\epsilon}\ln k)^{-1} + \sum_{k=1}^{K_0}(n_k \ln n_k)^{-1}.$$

But $\sum_{k=K_0+1}^{\infty}(k(\ln\ln k)^{1+\epsilon}\ln k)^{-1}$ converges by the integral test; hence $\sum_{k=1}^{\infty}(n_k \ln n_k)^{-1}$ converges and so does $\sum_{p\in S} p^{-1}$.

COROLLARY 6. *If for $k>K_0$, $n_k \leqq k \ln\ln k$ and $S=\{p_{n_k}\}_{k=1}^{\infty}$, then $\sum_{p\in S} p^{-1}$ diverges.*

Proof

$$\sum_{k=1}^{\infty}(n_k \ln n_k)^{-1} \geqq \sum_{k=K_0+1}^{\infty}(n_k \ln n_k)^{-1}$$
$$\geqq \sum_{k=K_0+1}^{\infty}(k\ln\ln k \ln(k\ln\ln k))^{-1}$$
$$\geqq \sum_{k=K_0+1}^{\infty}(k\ln\ln k(2\ln k))^{-1},$$

which diverges by the integral test.

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CONIC SECTIONS BY VECTOR METHODS

ARNOLD JOHNSON, University of Toledo

In the following we treat conic sections from a consistent vector viewpoint. This is rarely if ever done but since the advantages of a coordinate-free viewpoint are well known there seemed to be something to be gained and nothing to be lost by doing so.

Let V be a two-dimensional vector space with an inner product over the real field. Let $\mathcal{C}$ be a conic section in V with directrix $\mathcal{L}$, focus F and eccentricity e. Let E be a vector normal to $\mathcal{L}$ and set

$$\mathcal{L} = \{P: P\cdot E = a\}.$$

The line $\mathcal{L}$ divides the plane into two parts which it will be convenient to label "left" and "right" and we shall arbitrarily state that Q is in the left half-plane if $(Q\cdot E)-a<0$ and Q is in the right half-plane if $(Q\cdot E)-a>0$. Obviously, if $Q\cdot E=a$ then Q is on the line $\mathcal{L}$.

Let $d(Q, \mathcal{L})$ represent the distance of a point Q from $\mathcal{L}$. Then by the focus-directrix property

$$\mathcal{C} = \{Q: |Q - F| = ed(Q, \mathcal{L})\}.$$

Since

$$d(Q, \mathcal{L}) = \frac{|Q\cdot E - a|}{|E|}$$

it follows that

$$\mathcal{C} = \left\{Q: |Q - F| = e\frac{|Q\cdot E - a|}{|E|}\right\}.$$

Now the temptation is too great to resist to put $|E| = e$. Having done this we find

$$\mathcal{C} = \{Q: |Q - F| = |Q\cdot E - a|\} \tag{1}$$

which is then our basic equation for a conic section with directrix $\mathcal{L}$, focus F and eccentricity $|E| = e$. In case $e=0$ the conic reduces to

$$\mathcal{C} = \{Q: |Q - F| = |a|\}$$

which is a circle with center F and radius $|a|$. In this case the directrix has vanished or "moved to infinity."

Suppose now that V has the usual orthonormal basis vectors i and j and each vector P is represented by the column matrix $\binom{u}{v}$ where u is the x-coordinate $i\cdot P$ and v is the y-coordinate $j\cdot P$. Let the transpose of a matrix A be denoted by A^t. Squaring both sides of the equation $|Q-F| = |Q\cdot E-a|$ and writing $(Q\cdot E)^2 = (Q^tE)(E^tQ) = Q^t(EE^t)Q$ yields the quadratic form:

$$Q^t(I - EE^t)Q + 2(aE - F)^tQ + F^tF - a^2 = 0. \tag{2}$$

It follows immediately that the conic will be symmetrical about the origin iff $F=aE$.

Let us return to the basic equation $|Q-F| = |Q\cdot E-a|$ and impose the condition of symmetry $F=aE$. The conic is determined by the distance $d(F, \mathcal{L})$ of the focus from the directrix and by the eccentricity e. The condition $F=aE$ yields

$$d(F, \mathfrak{L}) = \frac{|F\cdot E - a|}{|E|} = \left|\frac{a}{e}(e^2 - 1)\right|.$$

If $e=1$ we have $d(F, \mathfrak{L})=0$; i.e., the focus is on the directrix. Otherwise $d(F, \mathfrak{L})$ can be made to have any fixed value by the choice of $|a|$. Hence, if $e\neq 1$ a conic section is symmetrical about a center.

Now let the conic be symmetrical about the origin and let $e<1$. In this case if a point Q on the conic and the focus F do not lie on the same side of the directrix we have

$$|Q - F| > d(Q, \mathfrak{L}) > ed(Q, \mathfrak{L})$$

which contradicts the definition of the conic $\mathfrak{C}$. Hence the points Q of the conic lie on one side of the directrix, say the left half-plane. Then $|Q\cdot E-a|=a-Q\cdot E$ and symmetry yields

$$\begin{aligned}|Q - F| + |Q - (-F)| &= |Q - F| + |Q+F| = |Q - F| + |-Q - F| \\ &= (a - Q\cdot E) + (a - (-Q\cdot E)) = 2a.\end{aligned}$$

Consequently, *the sum of the distances of any point on the conic to the symmetrical foci F and $-F$ is a constant,* which is a characterization of an ellipse.

Now suppose $e>1$ and that the conic is symmetrical about the origin. In this case $F=aE$ and

$$|F| = |a|e > \frac{|a|}{e} = d(0, \mathfrak{L});$$

consequently F is further from the origin than the directrix $\mathfrak{L}$. If a point Q on the conic is not in the half-plane including the focus it follows that

$$d(Q, \mathfrak{L}) + d(F, \mathfrak{L}) \leqq |Q - F| = ed(Q, \mathfrak{L})$$

whence

$$d(Q, \mathfrak{L}) \geqq \frac{d(F, \mathfrak{L})}{(e - 1)} = \frac{|a|}{e}\frac{(e^2 - 1)}{(e - 1)} = (e + 1)\frac{|a|}{e} > 2\frac{|a|}{e};$$

consequently $-Q$ is in the half-plane including F. Hence if Q lies on the conic then Q and $-Q$ are not in the same half-plane. Now suppose Q is in the left half-plane. Then $|Q\cdot E-a|=a-Q\cdot E$ and $|(-Q)\cdot E-a|=(-Q)\cdot E-a$, and symmetry yields

$$|Q - F| - |Q - (-F)| = (a - Q\cdot E) - ((-Q)\cdot E - a) = 2a.$$

That is to say *the difference of the distances of any point on the conic to the symmetrical foci F and $-F$ is a constant,* which identifies the conic as a hyperbola.

The polar form of the equation for a conic may be obtained very simply from equation (1). Let the focus be at the origin and let a point Q on the conic have polar coordinates (r, ϕ) and let E have polar coordinates (e, θ). Suppose Q and F

are in the right half-plane. Then $Q \cdot E - a > 0$ and $0 \cdot E - a > 0$, or $a < 0$ and consequently if d is the distance of the directrix from the origin we have $d = -a/e$. Hence $r = er \cos(\phi - \theta) - a$ or

$$r = \frac{-a}{1 - e\cos(\phi - \theta)} = \frac{ed}{1 - e\cos(\phi - \theta)}.$$

Changing Q and F to the left half-plane amounts to rotating E through an angle π, which yields

$$r = \frac{ed}{1 - e\cos(\phi - [\pi + \theta])} = \frac{ed}{1 + e\cos(\phi - \theta)}.$$

Let us return now to equation (2) and investigate its form. Let $E = \binom{s}{t}$ and $F = \binom{p}{q}$; then equation (2) yields

$$(3)\quad (x\ y)\begin{pmatrix} 1 - s^2 & -st \\ -st & 1 - t^2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + 2(as - p\ \ at - q)\begin{pmatrix} x \\ y \end{pmatrix} = a^2 - (p^2 + q^2).$$

Thus

$$\det(I - EE^t) = \begin{vmatrix} 1 - s^2 & -st \\ -st & 1 - t^2 \end{vmatrix} = 1 - (s^2 + t^2) = 1 - |E|^2 = 1 - e^2$$

Hence we have an ellipse only if det $(I - EE^t) > 0$, a parabola only if det $(I - EE^t) = 0$, and a hyperbola only if det $(I - EE^t) < 0$.

Suppose E is parallel to the positive x-axis. Then $E = \binom{e}{0}$, i.e., $s = e$ and $t = 0$. Equation (3) becomes

$$(x\ y)\begin{pmatrix} 1 - e^2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + 2(ae - p\ \ -q)\begin{pmatrix} x \\ y \end{pmatrix} = a^2 - (p^2 + q^2)$$

or

$$(4)\qquad (1 - e^2)x^2 + 2(ae - p)x + (y - q)^2 = a^2 - p^2.$$

The case $e = 1$ and $a = p$ yields the equation of the straight line $y = q$. The case $e = 1$ and $a \neq p$ yields

$$(y - q)^2 = 2(p - a)\left(x - \frac{(a + p)}{2}\right)$$

which is an equation of a parabola whose vertex is at

$$\begin{bmatrix} \dfrac{a + p}{2} \\ q \end{bmatrix},$$

i.e., half way between the directrix $\mathfrak{L}$ and the focus F. The directrix lies at a directed distance a from the origin and parallel to the y-axis. If $p=-a$, we have the familiar form $(y-q)^2=-4ax$ in which the vertex is on the y-axis.

In the case $e\neq 1$, $p=ae$ yields a considerable simplification in which both (ae, q) and $(-ae, q)$ are foci. Equation (4) then becomes

$$\frac{x^2}{a^2}+\frac{(y-q)^2}{a^2(1-e^2)}=1.$$

A given quadratic form

$$Q^tAQ+2Y^tQ+c=0$$

differs from (2) by a multiplicative constant k, i.e., we have

$$A=k(I-EE^t),\qquad Y=k(aE-F),\quad\text{and}\quad c=k(F^tF-a^2).$$

Set

$$E=e\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}.$$

Then $s=e\cos\theta$ and $t=e\sin\theta$ and

$$\cot 2\theta=\frac{\cos^2\theta-\sin^2\theta}{2\cos\theta\sin\theta}=\frac{s^2-t^2}{2st}.$$

To compute the cotangent of the angle that E makes with the positive x-axis let

$$A=\begin{pmatrix}\alpha & \beta\\ \beta & \gamma\end{pmatrix}.$$

Then

$$\begin{pmatrix}\alpha & \beta\\ \beta & \gamma\end{pmatrix}=k\begin{pmatrix}1-s^2 & -st\\ -st & 1-t^2\end{pmatrix}$$

and

$$\frac{s^2-t^2}{2st}=\frac{k[(1-s^2)-(1-t^2)]}{2k(-st)}=\frac{\alpha-\gamma}{2\beta},$$

whence $\cot 2\theta=(\alpha-\gamma)/2\beta$. It follows that the rotation

$$\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}x'\\ y'\end{pmatrix}=\begin{pmatrix}x\\ y\end{pmatrix}$$

will diagonalize the form $Q^tAQ+2Y^tQ+c=0$. Since $\det A=k^2\det(I-EE^t)$ we obtain the familiar test for the classification of quadratic forms according to whether $\det A=0$, >0, or <0.

A VARIATION OF FERMAT'S PROBLEM

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Introduction. In this paper we consider the problem of finding the value of

$$E(O, A, B) = \operatorname{Max}\{OP - AP - BP\},$$

where O, A, B are arbitrary fixed points and P is allowed to vary in the plane. This problem was proposed to the participants in the University of Kansas, Summer 1963, Undergraduate Research Program by Professor S. M. Shah. He considered earlier [2] a similar problem of finding

$$\operatorname*{Min}_{P}\{OP + AP + BP\}.$$

We first give a counterexample to the construction given by Courant and Robbins [**1**; pp. 354–359] for the equivalent problem of finding the point which minimizes $AP+BP-OP$.

The following results are obtained:

(i) If $\sphericalangle OAB \geqq 60°$, $\sphericalangle OBA \geqq 60°$, then the desired maximum occurs at the point P where $\sphericalangle OPA = \sphericalangle OPB = 60°$.

(ii) Otherwise the maximum occurs at A or B, whichever is farther from O.

(iii) If we let $OA=\alpha$, $OB=\beta$, $AB=\gamma$ and write $\sum\alpha^r=\alpha^r+\beta^r+\gamma^r$ and $\sum\alpha^2\beta^2=\alpha^2\beta^2+\alpha^2\gamma^2+\beta^2\gamma^2$, when (i) holds,

$$E(O, A, B) = \left\{\tfrac{1}{2}\left[\sum \alpha^2 - \left(6\sum \alpha^2\beta^2 - 3\sum \alpha^4\right)^{1/2}\right]\right\}^{1/2}$$

and when (ii) holds,

$$E(O, A, B) = \operatorname{Max}\{\alpha, \beta\} - \gamma.$$

In conclusion we state necessary and sufficient conditions for $E(O, A, B) > 0$ because of its application (cf. [**3**]).

Counterexample. The formal methods of mathematics sometimes reach out beyond one's original intention. For example, if the angle at C is greater than 120° the procedure of geometrical construction produces, instead of the solution P (which in this case is the point C itself, which gives $\operatorname{Min}_R\{AR+BR+CR\}$) another point P', from which the larger side AB of the triangle ABC appears under an angle of 120°, and the smaller sides under an angle of 60°. Certainly P' does not solve our minimum problem, but we may suspect that it has some relation to it. The answer is that P' solves the following problem: to minimize the expression $a+b-c$ ($a=AP$, $b=BP$, $c=CP$).

If the angles of a triangle ABC are all less than 120°, then the sum of the distances a, b, c from any point to A, B, C, respectively, is least at that point where each side of the triangle subtends an angle of 120°, and $a+b-c$ is least at vertex C, [**1**; p. 358].

That the solution proposed here by Courant and Robbins is false can be seen easily by considering the following simple examples:

(i) Triangle ABC an equilateral triangle. The proposed solution yields a minimum of $2a$. However, $a+b-c$ is zero at both A and B.

(ii) Triangle ABC a $20°-20°-140°$ triangle. The proposed solution yields a minimum of, say, $2a-PC$. However by reflection of P in AB one obtains a point Q such that obviously $AQ+BQ-CQ<2a-PC$ since $AP=AQ$, $BP=BQ$, but $CP<CQ$, (Fig. 1).

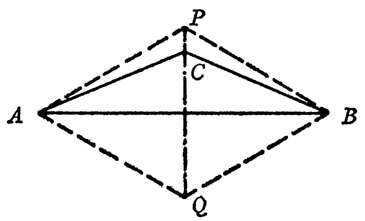

FIG. 1.

Solution.

I. $\mathrm{Max}_P\ \{OP-AP-BP\}$ must exist.

Since $OP_0-AP_0-BP_0$ is a continuous function of P_0 we need only examine $OP_0-AP_0-BP_0$ as, say $BP_0\to\infty$. But

$$OP_0 - AP_0 - BP_0 \leqq \underset{P}{\mathrm{Max}}\,(OP - AP) - BP_0 = OA - BP_0.$$

Since OA is fixed, $OP-AP-BP\to-\infty$ and is bounded above. Thus

$$\underset{P}{\mathrm{Max}}\,\{OP - AP - BP\}$$

is taken on at some point in the plane.

II. For the sake of calculations, let O be the origin of coordinates in the plane, A have co-ordinates (x_a, y_a) and B have co-ordinates (x_b, y_b). Then if an arbitrary point P has co-ordinates (x, y) define $F(P)=F(x, y)=OP-AP-BP$. Then

$$F(x, y) = [x^2 + y^2]^{1/2} - [(x - x_a)^2 + (y - y_a)^2]^{1/2} - [(x - x_b)^2 + (y - y_b)^2]^{1/2}$$

and

$$\frac{\partial F}{\partial x} = \frac{x}{[x^2 + y^2]^{1/2}} - \frac{x - x_a}{[(x - x_a)^2 + (y - y_a)^2]^{1/2}} - \frac{x - x_b}{[(x - x_b)^2 + (y - y_b)^2]^{1/2}}$$

$$\frac{\partial F}{\partial y} = \frac{y}{[x^2 + y^2]^{1/2}} - \frac{y - y_a}{[(x - x_a)^2 + (y - y_a)^2]^{1/2}} - \frac{y - y_b}{[(x - x_b)^2 + (y - y_b)^2]^{1/2}}.$$

$$\frac{\partial F}{\partial x} = 0 = \frac{\partial F}{\partial y}$$

is equivalent to

$$\left[\frac{\partial F}{\partial y}\right]^2 = \left[\frac{\partial F}{\partial y}\right]^2 = 0$$

or

$$\frac{x^2}{OP^2}+\frac{(x-x_a)^2}{AP^2}+\frac{(x-x_b)^2}{BP^2}-\frac{2x(x-x_a)}{OP\cdot AP}-\frac{2x(x-x_b)}{OP\cdot BP}+\frac{2(x-x_a)(x-x_b)}{AP\cdot BP}=0$$

and

$$\frac{y^2}{OP^2}+\frac{(y-y_a)^2}{AP^2}+\frac{(y-y_b)^2}{BP^2}-\frac{2y(y-y_a)}{OP\cdot AP}-\frac{2y(y-y_b)}{OP\cdot BP}+\frac{2(y-y_a)(y-y_b)}{AP\cdot BP}=0.$$

Addition yields

$$\frac{x^2+y^2}{OP^2}+\frac{(x-x_a)^2+(y-y_a)^2}{AP^2}+\frac{(x-x_b)^2+(y-y_b)^2}{BP^2}-2\,\frac{x(x-x_a)+y(y-y_a)}{OP\cdot AP}$$
$$-2\,\frac{x(x-x_b)+y(y-y_b)}{OP\cdot BP}+2\,\frac{(x-x_a)(x-x_b)+(y-y_a)(y-y_b)}{AP\cdot BP}=0$$

or

$$\frac{OP^2}{OP^2}+\frac{AP^2}{AP^2}+\frac{BP^2}{BP^2}-\frac{2OP\cdot AP\cos\sphericalangle APO}{OP\cdot AP}-\frac{2OP\cdot BP\cos\sphericalangle BPO}{OP\cdot BP}$$
$$+\frac{2AP\cdot BP\cos\sphericalangle APB}{AP\cdot BP}=0$$

and thus

$$3-2\cos\sphericalangle APO-2\cos\sphericalangle BPO+2\cos\sphericalangle APB=0.$$

If $\sphericalangle APO$ satisfies this equation, then $360°-\sphericalangle APO$ does and similarly for $\sphericalangle BPO$ and $\sphericalangle BPA$. So for an appropriate choice of angles, $\sphericalangle APB=\sphericalangle APO+\sphericalangle BPO$. Let $\sphericalangle APO=\theta$, $\sphericalangle BPO=\phi$. If θ and ϕ satisfy

$$3-2\cos\theta-2\cos\phi+2\cos(\theta+\phi)=0,$$

then

$$3-2\cos\theta-2\cos\phi+2\cos\theta\cos\phi=2\sin\theta\sin\phi,$$

which implies

$$[2\cos\theta(\cos\phi-1)+3-2\cos\phi]^2-4[1-\cos^2\theta][1-\cos^2\phi]=0,$$
$$\begin{aligned}0&=4\cos^2\theta(\cos\phi-1)^2+4\cos\theta(\cos\phi-1)(3-2\cos\phi)+9-12\cos\phi\\&\quad+4\cos^2\phi+4\cos^2\theta(1-\cos^2\phi)+4\cos^2\phi-4\\&=4\cos^2\theta(\cos\phi-1)(-2)+4\cos\theta(\cos\phi-1)(3-2\cos\phi)+8\cos^2\phi\\&\quad-12\cos\phi+5.\end{aligned}$$

Now $\cos\phi\neq 1$, for, if $\cos\phi=1$, then $0=8-12+5=1$. Hence

$$\cos\theta=\frac{4(1-\cos\phi)(3-2\cos\phi)}{16(1-\cos\phi)}$$

$$\pm \frac{[16(\cos\phi - 1)^2(3 - 2\cos\phi)^2 + 32(\cos\phi - 1)(8\cos^2\phi - 12\cos\phi + 5]^{1/2}}{16(1 - \cos\phi)}.$$

Since $\cos\theta$ is real,

$$\begin{aligned}0 &\leqq 16[\cos\phi-1][(\cos\phi-1)(9-12\cos\phi+4\cos^2\phi)+2(8\cos^2\phi-12\cos\phi+5)]\\ &=16[\cos\phi-1][9\cos\phi-12\cos^2\phi+4\cos^2\phi-9+12\cos\phi-4\cos^2\phi\\ &\qquad\qquad\qquad +16\cos^2\phi-24\cos\phi+10]\\ &=16(\cos\phi-1)(4\cos^2\phi-3\cos\phi+1)\\ &=16(\cos\phi-1)(\cos\phi+1)(2\cos\phi-1)^2\\ &=16(\cos^2\phi-1)(2\cos\phi-1)^2.\end{aligned}$$

However, this inequality does not hold unless $\cos\phi=1, -1$, or $\frac{1}{2}$. If $\cos\phi=-1$, then

$$3 - 2\cos\theta - 2\cos\phi + 2\cos(\theta+\phi) = 5 - 2(\cos\theta - \cos(\theta+\phi)) \geqq 1,$$

so it is necessary that $\cos\phi=\frac{1}{2}$. Then

$$\cos\theta = \frac{4(1-\frac{1}{2})(3-2\cdot\frac{1}{2})}{16(1-\frac{1}{2})} = \frac{1}{2}$$

and in fact $3-2\cdot\frac{1}{2}-2\cdot\frac{1}{2}+2(-\frac{1}{2})=0$. It follows that $\measuredangle APO=60°=\measuredangle BPO$ and $\measuredangle APB=120°$.

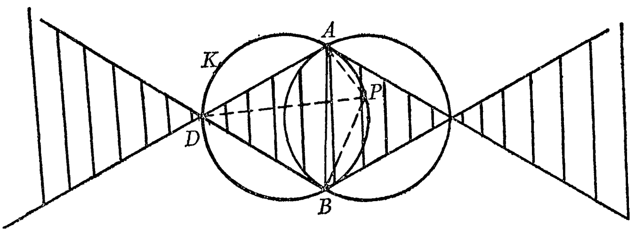

FIG. 2.

III. From elementary geometry, we know that the locus of points P such that $\measuredangle APB=120°$ is the minor arc AB of a circle K such that minor arc $AB=120°$. Obviously major arc $AB=240°$. Call the intersection of the perpendicular bisector of the segment AB with K, D (Fig. 2). Since

$$\text{major arc } AB = 240°, \text{ minor arc } AD = \text{minor arc } BD = 120°$$

so that $\measuredangle APD=\measuredangle DPB=60°$, and PD bisects $\measuredangle APB$. Thus when $\partial F/\partial x=\partial F/\partial y=0$ determines a solution, with the maximum given at P, O must lie on DP.

IV. It follows then that when O lies outside the shaded area in Figure 2, $\partial F/\partial x=0=\partial F/\partial y$ cannot determine a solution. This throws special importance on the fact that the partial derivatives do not exist at points O, A, and B. It remains then to compare $F(O)$, $F(A)$, $F(B)$ and, when P as above exists, $F(P)$.

Since $AB \leqq OA + OB$ and $OA > 0$, $F(O) = -OA - OB \leqq -AB < OA - AB = F(A)$. Thus, when O lies outside the shaded area in Figure 2,

$$\operatorname*{Max}_{P} \{OP - AP - BP\}$$

is given by A or B, whichever is farther from O.

V. When $O = D$, $F(A) = F(B) = F(P)$ for any point P on minor arc AB.

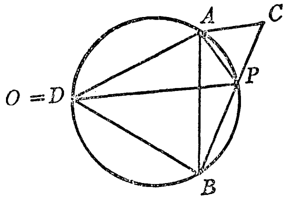

FIG. 3.

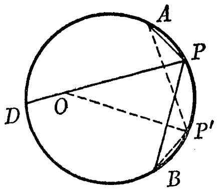

FIG. 4.

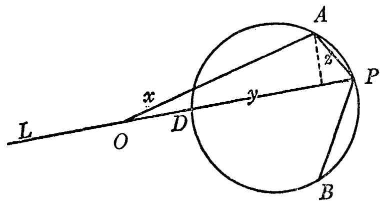

FIG. 5.

Proof. Triangle OAB is an equilateral triangle by construction when $O = D$, Figure 3. Therefore $F(A) = F(B) = 0$. Take an arbitrary point P, extend BP and construct $PC = AP$, as in Figure 3. Now since $\sphericalangle APB = 120°$, $\sphericalangle APC = 60°$. But triangle APC is isosceles so that $\sphericalangle PAC = \sphericalangle PCA = 60°$. In other words, $\sphericalangle OAP = \sphericalangle BAC$. But then $AB = OA$ and $AC = AP$ imply that triangle DPA is congruent to triangle BCA, and thus $BC = OP$. Hence $F(P) = OP - AP - BP = BC - AP - BP = BC - PC - BP = 0$.

VI. When O lies within triangle ABD and P is as above, $F(P)$ is not a maximum.

Proof. If P' is on arc AB (Fig. 4) then

$$F(P') = OP' - (AP' + BP') = OP' - DP'.$$

Now $OD + OP' > DP'$, and $F(P') = OP' - DP' > -OD = OP - DP = F(P)$. (That P is actually a saddle point in this case can be seen by noting that the difference $F(P) - F(P')$ for any fixed P' on the extended segment OP remains constant when we vary O on OP and by considering the following statement.)

VII. If O lies outside triangle ABD and there exists P as above, then

$$F(P) = \operatorname{Max} \{OP - AP - BP\}.$$

Proof. Let L be any line through D intersecting minor arc AB at P with, say $AP \leqq BP$ (Fig. 5). Let O range on L outside triangle ABD and call OP, y; OA, x; and AP, z. Then $x^2 = y^2 + z^2 - 2xy \cos 60°$. Hence

$$\frac{dx}{dy} = \frac{y - z \cos 60°}{x}.$$

But, $|y - z \cos 60°|$ is a leg of a right triangle of which x is the hypotenuse. Therefore,

$$\left|\frac{dx}{dy}\right| < 1 \quad \text{or} \quad |dx| < |dy|.$$

Then $F(P)=F(A)=0$ when $O=D$ implies that $F(P)>F(A)$ for O outside triangle ABD.

VIII. The complete solution is then the following:

For O in region TDU or $RD'S$, respectively, the point P on the further minor arc AB such that

$$\measuredangle APO = \measuredangle BPO = 60° \quad \text{yields} \quad F(P) = \text{Max}\{OP - AP - BP\}.$$

For O in the lower region bounded by U, DD', and S, $F(A)=\text{Max}\{OP-AP-BP\}$. And, for O in the upper region bounded by T, DD', and R, $F(B)=\text{Max}\{OP-AP-BP\}$ (Fig. 6).

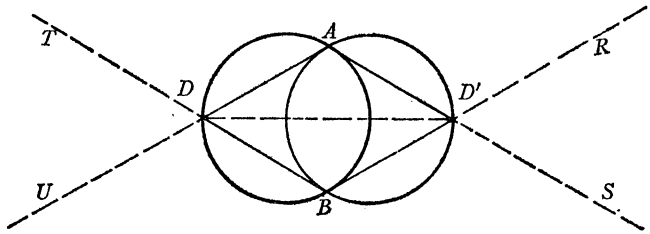

FIG. 6.

The algebraic value of $E(a, b)$. Let $OA=\alpha$, $OB=\beta$, $AB=\gamma$ and write $\sum\alpha^r=\alpha^r+\beta^r+\gamma^r$ and $\sum\alpha^2\beta^2=\alpha^2\beta^2+\beta^2\gamma^2+\alpha^2\gamma^2$. Then if P is determined by the construction above and (a) if $F(P)$ gives the maximum over the z-plane, then

$$E(O, A, B) = \left\{\frac{\sum\alpha^2 - (6\sum\alpha^2\beta^2 - 3\sum\alpha^4)^{1/2}}{2}\right\}^{1/2}$$

(b) if $F(A, B)=\text{Max}\{F(A), F(B)\}$ gives the maximum, then $E(O, A, B)=\max(\alpha, \beta)-\gamma$.

Proof of (a). Let $OP=\rho_1$, $AP=\rho_2$, and $BP=\rho_3$. Then

$$\alpha^2 = \rho_1^2 + \rho_2^2 - \rho_1\rho_2, \quad \text{since} \quad \measuredangle APO = 60°,$$
$$\beta^2 = \rho_1^2 + \rho_3^2 - \rho_1\rho_3, \quad \text{since} \quad \measuredangle BPO = 60°, \quad \text{and}$$
$$\gamma^2 = \rho_2^2 + \rho_3^2 + \rho_1\rho_3, \quad \text{since} \quad \measuredangle APB = 120°.$$

Define $t\equiv -F(P)=-\rho_1+\rho_2+\rho_3$ and $\eta\equiv\alpha^2+\beta^2+\gamma^2$.

A. $$\rho_1 = \frac{1}{3}\left(t + \frac{\eta - 3\alpha^2}{t}\right) \quad \text{and} \quad \rho_2 = \frac{1}{2}\left(t + \frac{\eta - 3\beta^2}{t}\right).$$

Then

$$\begin{aligned} 3\rho_3 t &= 3(\rho_2\rho_3 + \rho_3^2 - \rho_1\rho_3) \\ &= -3\rho_1\rho_3 + 3\rho_2\rho_3 + 3\rho_3^2 \end{aligned}$$

$$= 2(-\rho_1\rho_2 + \rho_2\rho_3 - \rho_1\rho_3) + (-\rho_1\rho_2 - \rho_1\rho_3 + \rho_2\rho_3) + 3\rho_3^2 + 3\rho_1\rho_2$$
$$= \rho_1^2 + \rho_2^2 + \rho_3^2 - 2\rho_1\rho_3 + 2\rho_2\rho_3 - 2\rho_1\rho_2 + \alpha^2 + \beta^2 + \gamma^2 - 3(\rho_1^2 + \rho_2^2 - \rho_1\rho_2)$$
$$= t^2 + \eta - 3\alpha^2.$$

Therefore

$$\rho_3 = \frac{1}{3}\left(t + \frac{\eta - 3\alpha^2}{t}\right).$$

Similarly,

$$\rho_2 = \frac{1}{3}\left(t + \frac{\eta - 3\beta^2}{t}\right).$$

B. Since $\gamma^2 = \rho_2^2 + \rho_3^2 + \rho_2\rho_3$, A implies that

$$9t^2\gamma^2 = 9t^2\left[\frac{1}{3}\left(t + \frac{\eta - 3\beta^2}{t}\right)\right]^2 + 9t^2\left[\frac{1}{3}\left(t + \frac{\eta - 3\alpha^2}{t}\right)\right]^2$$
$$+ 9t^2\left[\frac{1}{3}\left(t + \frac{\eta - 3\beta^2}{t}\right)\right]\left[\frac{1}{3}\left(t + \frac{\eta - 3\alpha^2}{t}\right)\right]$$
$$= [t^2 + \eta - 3\beta^2]^2 + [t^2 + \eta - 3\alpha^2]^2 + [t^2 + \eta - 3\alpha^2][t^2 + \eta - 3\beta^2].$$

C. Now B implies that

$$3t^2\gamma^2 = 3t^2(\eta - \alpha^2 - \beta^2) = t^4 + 2\eta t^2 - 3\alpha^2 t^2 - 3\beta^2 t^2 - 3\alpha^2\eta - 3\beta^2\eta + \eta^2$$
$$+3\alpha^4 + 3\beta^4 + 3\alpha^2\beta^2.$$

Hence

$$t^4 + t^2(2\eta - 3\eta) + (\eta^2 - 3\eta(\alpha^2 + \beta^2) + 3(\alpha^4 + \beta^2 + \alpha^2\beta^2)) = 0.$$
$$t^4 - t^2\eta + \alpha^4 + \beta^4 + \gamma^4 + 2\alpha^2\beta^2 + 2\alpha^2\gamma^2 + 2\beta^2\gamma^2 - 3\alpha^4 - 3\alpha^2\beta^2 - 3\alpha^2\gamma^2$$
$$- 3\alpha^2\beta^2 - 3\beta^4 - 3\beta^2\gamma^2 + 3\alpha^4 + 3\beta^4 + 3\alpha^2\sigma^2 = 0,$$

or

$$t^4 - \eta t^2 + \left(\sum \alpha^4 - \sum \alpha^2\beta^2\right) = 0.$$

Thus

$$t^2 = \frac{\eta \pm (\eta^2 - 4(\sum \alpha^4 - \sum \alpha^2\beta^2))^{1/2}}{2}$$
$$= \frac{\sum \alpha^2 \pm (6\sum \alpha^2\beta^2 - 3\sum \alpha^4)^{1/2}}{2}$$

and

$$2t^2 = \sum\alpha^2 \pm (6\sum \alpha^2\beta^2 - 3\sum \alpha^4)^{1/2}.$$

D. $\rho_2\rho_3<2\rho_2\rho_3+(\rho_2^2+\rho_3^2)=(\rho_2+\rho_3)^2$. But, as shown above, $\rho_1\geqq DP=\rho_2+\rho_3$. Therefore, $\rho_2\rho_3<\rho_1(\rho_2+\rho_3)$, and $-\rho_1\rho_2-\rho_1\rho_3+\rho_2\rho_3<0$. It then follows that

$$t^2 = \rho_1^2+\rho_2^2+\rho_3^2-2\rho_1\rho_2-2\rho_1\rho_3+2\rho_2\rho_3 < \rho_1^2+\rho_2^2+\rho_3^2-\rho_1\rho_2-\rho_1\rho_3+\rho_2\rho_3$$
$$< \rho_1^2+\rho_2^2+\rho_3^2\frac{-\rho_1\rho_2-\rho_1\rho_3+\rho_2\rho_3}{2} = \frac{1}{2}\sum\alpha^2.$$

Thus

$$t^2 < \frac{1}{2}\sum\alpha^2 \text{ implies } t^2 = \frac{\sum\alpha^2-(6\sum\alpha^2\beta^2-3\sum\alpha^4)^{1/2}}{2}.$$

Since we know $F(P)\geqq 0$,

$$-t = F(P) = \left\{\frac{\sum\alpha^2-(6\sum\alpha^2\beta^2-3\sum\alpha^4)^{1/2}}{2}\right\}^{1/2}.$$

Proof of (b).

$$\begin{aligned} F(A,B) &\equiv \max\{(OA-AB),(OB-AB)\} \\ &= \max(OA,OB)-AB \\ &= \max(\alpha,\beta)-\gamma. \end{aligned}$$

The sign of the maximum. In the interest of easy application we consider the problem in its complex number form with $O=0$ (zero), $A=a$, $B=b$.

I. (A) $\operatorname{Max}\{OP-AP-BP\}=0$, when $\max\{|a|,|b|\}=|a-b|$.
(B) $\operatorname{Max}\{OP-AP-BP\}>0$, when $\max\{|a|,|b|\}>|a-b|$.
(C) $\operatorname{Max}\{OP-AP-BP\}<0$, when $\max\{|a|,|b|\}<|a-b|$.

Proof of A. We have shown above that $F(P)=\max\{OP-AP-BP\}=0$ only when $|a|=|b|=|a-b|$. Therefore let us consider the case when $F(A,B)=\max\{OP-AP-BP\}=0$. Then $P=A\,[B]$ yields the desired maximum where $|a|\geqq|b|\;[|b|\geqq|a|]$. Obviously $F(A,B)=0$ when

(i) $|b|\geqq|a|,\quad |b|=|a-b|$, or
(ii) $|a|\geqq|b|,\quad |a|=|a-b|$.

Proof of B. When $O=D$, $|a|=|b|=|a-b|$ and $F(P)=F(A,B)=0$. But when $\max\{|a|,|b|\}>|a-b|$, $\max\{OP-AP-BP\}\geqq\max\{|a|,|b|\}-|a-b|>0$.

Proof of C. We have shown above that $F(P)\geqq 0$. Therefore we need only consider O such that $F(A,B)$ yields the desired maximum. But

$$F(A,B)=\max\{|a|,|b|\}-|a-b|<0,$$

when $\max\{|a|,|b|\}<|a-b|$.

Note. The geometrical interpretation is that

(A) holds when O lies on one of the darkened arcs

(B) holds when O lies outside the darkened arcs, and
(C) holds when O lies within the darkened arcs (Fig. 7).

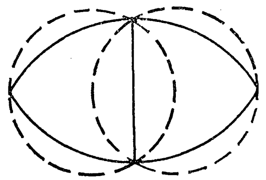

FIG. 7.

II. Argument conditions for the sign of the maximum of the function.

$$|\arg a - \arg b| \leqq \cos^{-1}\left[\frac{\operatorname{Min}\{|a|, |b|\}}{2\operatorname{Max}\{|a|, |b|\}}\right] \quad \text{for } |\arg a - \arg b| \leqq \pi,$$

$$\operatorname{Max}\{|z| - |z-a| - |z-b|\} \geqq 0 \text{ iff } 2\pi - |\arg a - \arg b|$$

$$\leqq \cos^{-1}\left[\frac{\operatorname{Min}\{|a|, |b|\}}{2\operatorname{Max}\{|a|, |b|\}}\right] \quad \text{for } |\arg a - \arg b| > \pi$$

with the maximum equal to zero iff the equalities hold.

Proof. Let θ be the angle determined by the lines from the origin to the complex numbers a and b. That is, if $|\arg a-\arg b| \leqq \pi$, $\theta = |\arg a - \arg b|$, and if $|\arg a-\arg b| > \pi$, $\theta = 2\pi - |\arg a - \arg b|$.

Suppose $|b| \geqq |a|$. Then letting $z=b$, we have $|z|-|z-a|-|z-b| = |b| - |a-b|$. Thus the function evaluated at b is greater than or equal to zero iff $|b| \geqq |a-b|$. Now by the law of cosines, $|a-b|^2 = |a|^2 + |b|^2 - 2|a||b| \cos\theta$, so the above condition becomes $|b|^2 \geqq |a|^2 + |b|^2 - 2|a||b| \cos\theta$, or $2|a||b| \cos\theta \geqq |a|^2$ or equivalently $\cos\theta \geqq |a|/2|b|$.

Now since $\theta \leqq \pi$, it is necessary that $0 \leqq \theta < \pi/2$ for the inequality to hold. However, the cosine function is decreasing on the interval $[0, \pi/2]$ so that the function evaluated at b is greater than or equal to zero if $f\theta \leqq \cos^{-1}[|a|/2|b|]$.

If the inequality does not hold, since $|b| \geqq |a|$ we know that $\theta > \pi/3$ and from the geometric solution (statement VI) the maximum of the function then occurs at b so that it is less than zero.

If the equality holds we know that $|b| = |a-b|$. But from the modulus conditions above, if $\operatorname{Max}\{|a|, |b|\} = |a-b|$, the maximum is zero and the result follows.

We notice from the proof above that the function evaluated at the fixed point farthest from the origin always has the same sign as the maximum value. Also, if the angle at the origin is less than $\pi/3$, the maximum is positive.

The authors would like to express their thanks for the kind assistance of Professor Shah and Professor George Springer in the preparation of this paper. We are also indebted to the referee for many suggestions which make the discussion of the problem clearer and for the correct origin of the problem.

This work was supported by a National Science Foundation Undergraduate Science Education Program, GE 1937.

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3. ———, On the order of the difference of two meromorphic functions, Proc. Amer. Math. Soc., 12 (1961) 234–242.

SOME RADICAL AXES ASSOCIATED WITH THE CIRCUMCIRCLE

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PART 2

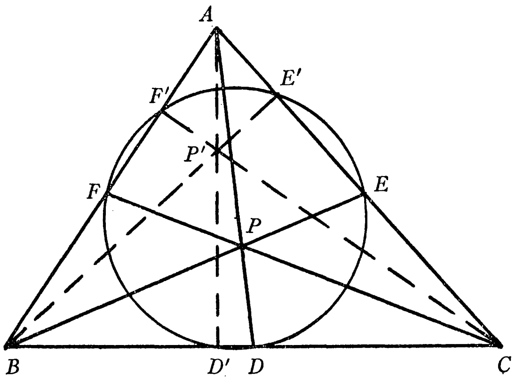

FIG. 3.

Construct circle DEF, the circumcircle of the cevian triangle of point P, and let this circle meet sides BC, CA, AB again at points D', E', F' respectively (Fig. 3). Rays AD', BE', CF' can be shown to be concurrent and it is known that

$$\frac{BD'}{D'C}=\frac{a^2\left(\frac{BF}{FA}+1\right)\left(\frac{AE}{EC}+1\right)-b^2\left(\frac{BD}{DC}+1\right)\left(\frac{BF}{FA}+1\right)+c^2\left(\frac{BD}{DC}+1\right)\left(\frac{CE}{EA}+1\right)}{a^2\left(\frac{BF}{FA}+1\right)\left(\frac{AE}{EC}+1\right)+b^2\left(\frac{BD}{DC}+1\right)\left(\frac{BF}{FA}+1\right)-c^2\left(\frac{BD}{DC}+1\right)\left(\frac{CE}{EA}+1\right)},$$

with similar values existing for ratios $CE'/E'A$ and $AF'/F'B$ [5, Theorem 2]. The three products in the denominator of $BD'/D'C$ may be rewritten as

$$\begin{aligned}a^2\left(\frac{BF}{FA}+1\right)\left(\frac{AE}{EC}+1\right)&=a^2\left(\frac{AF}{FB}+1\right)\left(\frac{CE}{EA}+1\right)\frac{BF}{FA}\cdot\frac{AE}{EC}\\&=a^2\left(\frac{AF}{FB}+1\right)\left(\frac{CE}{EA}+1\right)\frac{BD}{DC},\end{aligned}$$

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References

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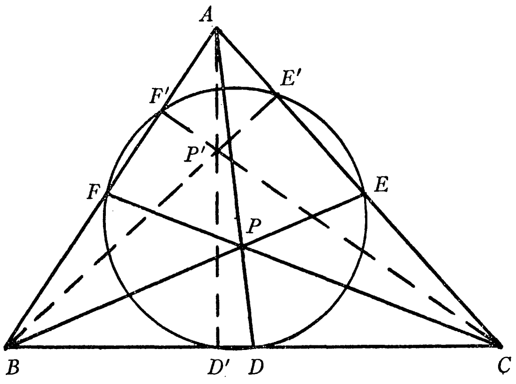

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$$\begin{aligned} a^2\left(\frac{BF}{FA}+1\right)\left(\frac{AE}{EC}+1\right) &= a^2\left(\frac{AF}{FB}+1\right)\left(\frac{CE}{EA}+1\right)\frac{BF}{FA}\cdot\frac{AE}{EC} \\ &= a^2\left(\frac{AF}{FB}+1\right)\left(\frac{CE}{EA}+1\right)\frac{BD}{DC}, \end{aligned}$$

$$b^2\left(\frac{BD}{DC}+1\right)\left(\frac{BF}{FA}+1\right)=b^2\left(\frac{CD}{DB}+1\right)\left(\frac{BF}{FA}+1\right)\frac{BD}{DC}, \text{ and}$$

$$c^2\left(\frac{BD}{DC}+1\right)\left(\frac{CE}{EA}+1\right)=c^2\left(\frac{CD}{BD}+1\right)\left(\frac{CE}{EA}+1\right)\frac{BD}{DC}.$$

It thus appears that the ratio BD/DC is a factor of the denominator of $BD'/D'C$.

If MNO be the radical axis of circle DEF and circumcircle ABC, it is known from Theorem 1 that $BM/MC=-(BD/DC)(BD'/D'C)$. By rewriting the denominator of $BD'/D'C$, as indicated above, it is found that

$$\frac{BM}{MC}=-\frac{a^2\left(\frac{BF}{FA}+1\right)\left(\frac{AE}{EC}+1\right)-b^2\left(\frac{BD}{DC}+1\right)\left(\frac{BF}{FA}+1\right)+c^2\left(\frac{BD}{DC}+1\right)\left(\frac{CE}{EA}+1\right)}{a^2\left(\frac{AF}{FB}+1\right)\left(\frac{CE}{EA}+1\right)+b^2\left(\frac{CD}{DB}+1\right)\left(\frac{BF}{FA}+1\right)-c^2\left(\frac{CD}{DB}+1\right)\left(\frac{CE}{EA}+1\right)}.$$

Ratios CN/NA and AO/OB are found in the same manner.

THEOREM 6. *Let P be any point in the plane of triangle ABC, with DEF its cevian triangle. The radical axis of circle DEF and circumcircle ABC meets sides BC, CA, AB at respective points M, N, O, so that*

$$\frac{BM}{MC}=-\frac{a^2\left(\frac{BF}{FA}+1\right)\left(\frac{AE}{EC}+1\right)-b^2\left(\frac{BD}{DC}+1\right)\left(\frac{BF}{FA}+1\right)+c^2\left(\frac{BD}{DC}+1\right)\left(\frac{CE}{EA}+1\right)}{a^2\left(\frac{AF}{FB}+1\right)\left(\frac{CE}{EA}+1\right)+b^2\left(\frac{CD}{DB}+1\right)\left(\frac{BF}{FA}+1\right)-c^2\left(\frac{CD}{DB}+1\right)\left(\frac{CE}{EA}+1\right)},$$

$$\frac{CN}{NA}=-\frac{b^2\left(\frac{CD}{DB}+1\right)\left(\frac{BF}{FA}+1\right)-c^2\left(\frac{CE}{EA}+1\right)\left(\frac{CD}{DB}+1\right)+a^2\left(\frac{CE}{EA}+1\right)\left(\frac{AF}{FB}+1\right)}{b^2\left(\frac{BD}{DC}+1\right)\left(\frac{AF}{FB}+1\right)+c^2\left(\frac{AE}{EC}+1\right)\left(\frac{CD}{DB}+1\right)-a^2\left(\frac{AE}{EC}+1\right)\left(\frac{AF}{FB}+1\right)},$$

$$\frac{AO}{OB}=-\frac{c^2\left(\frac{AE}{EC}+1\right)\left(\frac{CD}{DB}+1\right)-a^2\left(\frac{AF}{FB}+1\right)\left(\frac{AE}{EC}+1\right)+b^2\left(\frac{AF}{FB}+1\right)\left(\frac{BD}{DC}+1\right)}{c^2\left(\frac{CE}{EA}+1\right)\left(\frac{BD}{DC}+1\right)+a^2\left(\frac{BF}{FA}+1\right)\left(\frac{AE}{EC}+1\right)-b^2\left(\frac{BF}{FA}+1\right)\left(\frac{BD}{DC}+1\right)}.$$

Let P be the centroid of triangle ABC so that $BD/DC=CE/EA=AF/FB=1$. Circle DEF then becomes the nine point circle of triangle ABC and Theorem 6 yields

$$\frac{BM}{MC}=-\frac{a^2+c^2-b^2}{a^2+b^2-c^2},\quad \frac{CN}{NA}=-\frac{a^2+b^2-c^2}{b^2+c^2-a^2},\quad \frac{AO}{OB}=-\frac{b^2+c^2-a^2}{a^2+c^2-b^2}.$$

From the ratios listed immediately following the statement of Theorem 2, it is apparent that the radical axis of the nine point circle and the circumcircle is the trilinear polar of the orthocenter of triangle ABC. In this instance, MNO is also called the orthic axis of the triangle.

THEOREM 6A. *The orthic axis is the radical axis of the nine point circle and circumcircle of the triangle.*

Suppose that P is now the Gergonne point of triangle ABC. Points D, E, F become the points of contact of the incircle of the triangle. Furthermore, the ratio values for the Gergonne point are known to be

$$\frac{BD}{DC}=\frac{a+c-b}{a+b-c},\quad \frac{CE}{EA}=\frac{a+b-c}{b+c-a},\quad \frac{AF}{FB}=\frac{b+c-a}{a+c-b}.$$

These values substituted in Theorem 6 determine the radical axis of the incircle and circumcircle. However, we may consider the incircle as a special case of circle (D, D') of Theorem 1 for which points D and D' coincide. Ratios BD/DC and $BD'/D'C$ are then identical and

$$\frac{BM}{MC}=-\frac{BD}{DC}\cdot\frac{BD'}{D'C}=-\left(\frac{BD}{DC}\right)^2=-\left(\frac{a+c-b}{a+b-c}\right)^2.$$

In a similar manner points E and E' on side CA coincide as do points F and F' on side AB. That the following result is correct may be verified by substituting the ratio values for the Gergonne point in Theorem 6.

THEOREM 6B. *The radical axis of the incircle and circumcircle of triangle ABC meets sides BC, CA, AB at respective points M, N, O, so that*

$$\frac{BM}{MC}=-\left(\frac{a+c-b}{a+b-c}\right)^2,\quad \frac{CN}{NA}=-\left(\frac{a+b-c}{b+c-a}\right)^2,\quad \frac{AO}{OB}=-\left(\frac{b+c-a}{a+c-b}\right)^2.$$

If P be any point in the plane of triangle ABC and DEF its cevian triangle, sides EF, FD, DE may be extended to meet BC, CA, AB at respective points D', E', F'. Let M, N, O be the midpoints of segments DD', EE', FF'. It is shown in numerous geometry texts that $BM/MC=-(BD/DC)^2$, $CN/NA=-(CE/EA)^2$, $AO/OB=-(AF/FB)^2$ and points M, N, O are collinear. When P is the Gergonne point, it is evident from the preceding theorem that the midpoints of segments DD', EE', FF' determine the radical axis of the incircle and circumcircle. So these considerations provide a method for constructing this radical axis.

Let the excircle relative to vertex A touch sides BC, CA, AB at points D, E, F respectively. Rays AD, BE, CF are then concurrent at point P and

$$\frac{BD}{DC}=\frac{a+b-c}{a+c-b},\quad \frac{CE}{EA}=-\frac{a+c-b}{a+b+c},\quad \frac{AF}{FB}=-\frac{a+b+c}{a+b-c}.$$

The use of Theorem 1, or the more lengthy substitution in Theorem 6, determines the desired axis.

THEOREM 6C. *The radical axis of the excircle relative to vertex A and the circumcircle of triangle ABC meets sides BC, CA, AB at points M, N, O, so that*

$$\frac{BM}{MC}=-\left(\frac{a+b-c}{a+c-b}\right)^2,\quad \frac{CN}{NA}=-\left(\frac{a+c-b}{a+b+c}\right)^2,\quad \frac{AO}{OB}=-\left(\frac{a+b+c}{a+b-c}\right)^2.$$

Ratio values for the points of contact of the excircle relative to vertex B are

$$\frac{BD}{DC} = -\frac{a+b+c}{b+c-a}, \quad \frac{CE}{EA} = \frac{b+c-a}{a+b-c}, \quad \frac{AF}{FB} = -\frac{a+b-c}{a+b+c}.$$

The radical axis of this circle and the circumcircle determines the ratios

$$\frac{BM}{MC} = -\left(\frac{a+b+c}{b+c-a}\right)^2, \quad \frac{CN}{NA} = -\left(\frac{b+c-a}{a+b-c}\right)^2, \quad \frac{AO}{OB} = -\left(\frac{a+b-c}{a+b+c}\right)^2.$$

Finally, the excircle relative to vertex C touches the triangle sides so that

$$\frac{BD}{DC} = -\frac{b+c-a}{a+b+c}, \quad \frac{CE}{EA} = -\frac{a+b+c}{a+c-b}, \quad \frac{AF}{FB} = \frac{a+c-b}{b+c-a}.$$

Radical axis MNO is fixed by the ratios

$$\frac{BM}{MC} = -\left(\frac{b+c-a}{a+b+c}\right)^2, \quad \frac{CN}{NA} = -\left(\frac{a+b+c}{a+c-b}\right)^2, \quad \frac{AO}{OB} = -\left(\frac{a+c-b}{b+c-a}\right)^2.$$

The reader should observe that the ratios associated with the radical axes mentioned in Theorems 6B and 6C are all negative. This means that these four axes always cut the sides of triangle ABC externally.

The reader may allow P to be the incenter, circumcenter, symmedian point, either of the Brocard points, etc., of triangle ABC. The ratios associated with the radical axis of the circumcircles of the cevian triangle of point P and triangle ABC may then be computed through the use of Theorem 6.

Let two straight lines in the plane of triangle ABC meet sides BC, CA, AB at points M, N, O and M', N', O' respectively. Suppose that lines MNO and $M'N'O'$ meet at point P and extend rays AP, BP, CP to meet sides BC, CA, AB at respective points D, E, F. It is then known [6] that

$$\frac{BD}{DC} = -\frac{BO/OA - BO'/O'A}{CN/NA - CN'/N'A}, \quad \frac{CE}{EA} = -\frac{CM/MB - CM'/M'B}{AO/OB - AO'/O'B},$$

$$\frac{AF}{FB} = -\frac{AN/NC - AN'/N'C}{BM/MC - BM'/M'C}.$$

Allow lines MNO and $M'N'O'$ to be the radical axes of Theorems 6A and 6B. Point P then becomes the point of intersection of these radical axes and is therefore the radical center of the incircle, nine point circle, and circumcircle. The ratio values for BM/MC, CN/NA, AO/OB are given in the paragraph preceding the statement of Theorem 6A. Those for $BM'/M'C$, $CN'/N'A$, $AO'/O'B$ are found in Theorem 6B. Substitution shows that

$$\begin{aligned}\frac{BD}{DC} &= -\frac{BO/OA - BO'/O'A}{CN/NA - CN'/N'A}\\ &= \left(\frac{a-b}{c-a}\right)\left(\frac{a^3+b^3-2c^3+ac^2+bc^2-ab^2-a^2b}{a^3+c^3-2b^3+b^2c+ab^2-a^2c-ac^2}\right).\end{aligned}$$

CE/EA and AF/FB are derived in similar fashion.

THEOREM 7. *Point P is the radical center of the incircle, nine point circle, and circumcircle of triangle ABC, with DEF its cevian triangle. The ratio values for point P are*

$$\frac{BD}{DC}=\left(\frac{a-b}{c-a}\right)\left(\frac{a^3+b^3-2c^3+ac^2+bc^2-ab^2-a^2b}{a^3+c^3-2b^3+b^2c+ab^2-a^2c-ac^2}\right),$$
$$\frac{CE}{EA}=\left(\frac{b-c}{a-b}\right)\left(\frac{b^3+c^3-2a^3+a^2b+a^2c-bc^2-b^2c}{a^3+b^3-2c^3+ac^2+bc^2-ab^2-a^2b}\right),$$
$$\frac{AF}{FB}=\left(\frac{c-a}{b-c}\right)\left(\frac{a^3+c^3-2b^3+b^2c+ab^2-a^2c-ac^2}{b^3+c^3-2a^3+a^2b+a^2c-bc^2-b^2c}\right).$$

When P and P' are any two points in the plane of triangle ABC, with DEF and $D'E'F'$ their cevian triangles, it is evident that the radical axis of circles DEF and ABC may be determined by the use of Theorem 6. The radical axis of circles $D'E'F'$ and ABC is obtained in the same fashion. We may then proceed, as we did prior to the statement of Theorem 7, and determine the radical center of circles DEF, $D'E'F'$, ABC.

Let P and P' again be a pair of isogonal conjugates in triangle ABC, with DEF and $D'E'F'$ their respective cevian triangles. From point P drop perpendiculars to sides BC, CA, AB, thereby determining respective points G, H, I. In similar fashion perpendiculars from P' to the sides of the triangle fix points G', H', I'. Triangle GHI ($G'H'I'$) is called the pedal triangle of point P (P') with respect to triangle ABC and circle GHI ($G'H'I'$) is known as the pedal circle of point P (P'). A well-known geometrical theorem states that the six points G, H, I, G', H', I' are cyclic with the center of the pedal circle lying at the midpoint of segment PP'.

Now

$$\frac{BG}{GC}=\frac{2a^2(AE/EC)+a^2+c^2-b^2}{2a^2(AF/FB)+a^2+b^2-c^2}\quad\text{and}\quad\frac{BG'}{G'C}=\frac{2a^2(AE'/E'C)+a^2+c^2-b^2}{2a^2(AF'/F'B)+a^2+b^2-c^2},$$

with similar expressions existing for ratios CH/HA, $CH'/H'A$, AI/IB, $AI'/I'B$ [7]. In ratio $BG'/G'C$, $AE'/E'C$ and $AF'/F'B$ may be replaced by $(c^2/a^2)(CE/EA)$ and $(b^2/a^2)(BF/FA)$, since the equations $BD/DC\cdot BD'/D'C=c^2/b^2$, $CE/EA\cdot CE'/E'A=a^2/c^2$, $AF/FB\cdot AF'/F'B=b^2/a^2$ are always true for a pair of isogonal conjugates. We then have

$$\frac{BG'}{G'C}=\frac{2a^2(AE'/E'C)+a^2+c^2-b^2}{2a^2(AF'/F'B)+a^2+b^2-c^2}=\frac{2c^2(CE/EA)+a^2+c^2-b^2}{2b^2(BF/FA)+a^2+b^2-c^2}.$$

Theorem 1 then shows that the radical axis of pedal circle GHI ($G'H'I'$) and circumcircle ABC meets BC at M so that

$$\frac{BM}{MC}=-\frac{BG}{GC}\cdot\frac{BG'}{G'C}$$

$$= -\left(\frac{2a^2(AE/EC) + a^2 + c^2 - b^2}{2a^2(AF/FB) + a^2 + b^2 - c^2}\right)\left(\frac{2c^2(CE/EA) + a^2 + c^2 - b^2}{2b^2(BF/FA) + a^2 + b^2 - c^2}\right).$$

Sides CA and AB are treated in the same manner to obtain

THEOREM 8. *Let P be any point in the plane of triangle ABC, with DEF its cevian triangle. The radical axis of the pedal circle of point P and circumcircle ABC meets sides BC, CA, AB at respective points M, N, O, so that*

$$\frac{BM}{MC} = -\left(\frac{2a^2(AE/EC) + a^2 + c^2 - b^2}{2a^2(AF/FB) + a^2 + b^2 - c^2}\right)\left(\frac{2c^2(CE/EA) + a^2 + c^2 - b^2}{2b^2(BF/FA) + a^2 + b^2 - c^2}\right),$$

$$\frac{CN}{NA} = -\left(\frac{2b^2(BF/FA) + a^2 + b^2 - c^2}{2b^2(BD/DC) + b^2 + c^2 - a^2}\right)\left(\frac{2a^2(AF/FB) + a^2 + b^2 - c^2}{2c^2(CD/DB) + b^2 + c^2 - a^2}\right),$$

$$\frac{AO}{OB} = -\left(\frac{2c^2(CD/DB) + b^2 + c^2 - a^2}{2c^2(CE/EA) + a^2 + c^2 - b^2}\right)\left(\frac{2b^2(BD/DC) + b^2 + c^2 - a^2}{2a^2(AE/EC) + a^2 + c^2 - b^2}\right).$$

Let ratio values

$$\frac{BD}{DC} = \frac{a^2 + c^2 - b^2}{a^2 + b^2 - c^2}, \quad \frac{CE}{EA} = \frac{a^2 + b^2 - c^2}{b^2 + c^2 - a^2}, \quad \frac{AF}{FB} = \frac{b^2 + c^2 - a^2}{a^2 + c^2 - b^2}$$

associated with the cevian triangle of the orthocenter be substituted in the equations of Theorem 8. Without too much difficulty the first factor in the right member of ratio BM/MC is found to have the value

$$\frac{a^2 + c^2 - b^2}{a^2 + b^2 - c^2}.$$

The second factor rather easily reduces to $+1$. In this way the results given in Theorem 6A are obtained. However, the reader will observe that Theorem 6 and the ratios connected with the centroid afford an easier method of determining the radical axis of the nine point circle and circumcircle than does Theorem 8 and the cevian ratios associated with the orthocenter.

Again, let the incenter cevian ratios $BD/DC = c/b$, $CE/EA = a/c$, $AF/FB = b/a$ be placed in the equations of Theorem 8. This time the conclusions of Theorem 6B are obtained. As before, the result is easier to secure through the reasoning preceding the statement of Theorem 6B than through the use of Theorem 8.

Suppose that P be the centroid of triangle ABC so that $BD/DC = CE/EA = AF/FB = 1$. Theorem 8 then becomes

THEOREM 8A. *The radical axis of the pedal circle of the centroid (symmedian point) and circumcircle ABC meets sides BC, CA, AB at respective points M, N, O so that*

$$\frac{BM}{MC} = -\left(\frac{3a^2 + c^2 - b^2}{3a^2 + b^2 - c^2}\right)\left(\frac{3c^2 + a^2 - b^2}{3b^2 + a^2 - c^2}\right),$$

$$\frac{CN}{NA} = -\left(\frac{3b^2 + a^2 - c^2}{3b^2 + c^2 - a^2}\right)\left(\frac{3a^2 + b^2 - c^2}{3c^2 + b^2 - a^2}\right),$$

$$\frac{AO}{OB} = -\left(\frac{3c^2 + b^2 - a^2}{3c^2 + a^2 - b^2}\right)\left(\frac{3b^2 + c^2 - a^2}{3a^2 + c^2 - b^2}\right).$$

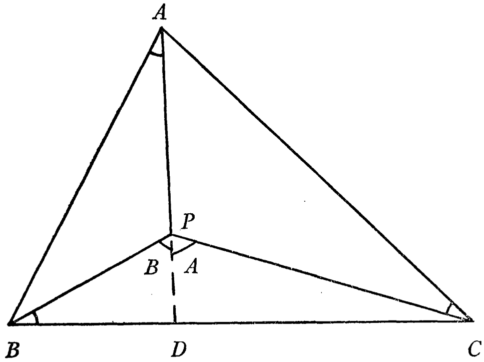

FIG. 4.

Let P be a point in the plane of triangle ABC such that $\angle PAB = \angle PBC = \angle PCA$ (Fig. 4). Point P is known as the positive Brocard point of the triangle and P', its isogonal conjugate, is called the negative Brocard point of the triangle. Extend AP to meet BC at D. It is then evident that $\angle BPD = \angle B$ and $\angle DPC = \angle A$. Angle APB and angle B are supplementary and so the law of sines applied to triangle PAB yields $BP/c = (\sin\ \angle PAB)/(\sin\ \angle B)$ or (1) $\sin\ \angle PAB = (BP\cdot\sin\ \angle B)/c$. In similar fashion triangle PBC gives (2) $\sin\ \angle PBC = (PC\cdot\sin\ \angle C)/a$. Since angles PAB and PBC are equal, the right members of equations (1) and (2) may be placed equal to each other. The resulting equation yields $BP/PC = (c\cdot\sin\ \angle C)/(a\cdot\sin\ \angle B) = c^2/ab$.

Application of the law of sines to triangles PBD, PDC, PBC gives $BD/DC = (BP/PC)(\sin\ \angle BPD/\sin\ \angle DPC) = (BP/PC)(\sin\ \angle B/\sin\ \angle A)$. Replacing BP/PC by the value determined at the end of the preceding paragraph, and substituting b/a for $\sin\ \angle B/\sin\ \angle A$, this becomes $BD/DC = c^2/a^2$. In the same way ratios CE/EA and AF/FB are calculated and we find that the ratios for the positive Brocard point are $BD/DC = c^2/a^2$, $CE/EA = a^2/b^2$, $AF/FB = b^2/c^2$. In like manner the negative Brocard point P' has ratio values $BD'/D'C = a^2/b^2$, $CE'/E'A = b^2/c^2$, $AF'/F'B = c^2/a^2$. The ratio values for the positive Brocard point substituted in Theorem 8 give

THEOREM 8B. *The radical axis of the pedal circle of the positive (negative) Brocard point and circumcircle ABC meets sides BC, CA, AB at respective points M, N, O so that*

$$\frac{BM}{MC} = -\frac{c^2}{b^2}\left(\frac{2a^2c^2 + a^2b^2 + b^2c^2 - b^4}{2a^2b^2 + a^2c^2 + b^2c^2 - c^4}\right),$$

$$\frac{CN}{NA} = -\frac{a^2}{c^2}\left(\frac{2a^2b^2 + b^2c^2 + a^2c^2 - c^4}{2b^2c^2 + a^2b^2 + a^2c^2 - a^4}\right),$$

$$\frac{AO}{OB} = -\frac{b^2}{a^2}\left(\frac{2b^2c^2 + a^2c^2 + a^2b^2 - a^4}{2a^2c^2 + b^2c^2 + a^2b^2 - b^4}\right).$$

If the cevian ratios for the negative Brocard point are placed in the equations of Theorem 8, the results of Theorem 8B will be obtained. So generally, when determining the radical axis of the circumcircle and pedal circle, ratio values for either of the isogonal conjugates may be used in Theorem 8.

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A METHOD FOR FINDING THE SOLUTION OF A NON-HOMOGENEOUS DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS

A. A. HOOMANI and J. W. BYRD, East Carolina College

Introduction. There are several well-known techniques for generating the particular solution of a differential equation with constant coefficients. Among these, the method of undetermined coefficients is often used. The analysis presented in this paper is applicable, in principle, to more general equations than those solvable by undetermined coefficients although it has been applied successfully only to this class of equations. The method of undetermined coefficients is simple as long as the right side of the equation is a sum of polynomials, sines, cosines, and exponentials. However, when the right side is a product of these functions, the trial solution is difficult to find without a good deal of experience or a reference text such as [2]. The method developed in the paper eliminates the need for such trial solutions.

General theory. Consider a linear, n-th order differential equation given by

$$f(D)y = R(x), \tag{1}$$

where $f(D)$ is a differential operator of order n. Now, choose an operator $g(D)$ such that

$$g(D)R(x) = 0. \tag{2}$$

Operating on (1) with this operator we obtain

$$g(D)f(D)y = 0. \tag{3}$$

$$\frac{CN}{NA} = -\frac{a^2}{c^2}\left(\frac{2a^2b^2 + b^2c^2 + a^2c^2 - c^4}{2b^2c^2 + a^2b^2 + a^2c^2 - a^4}\right),$$

$$\frac{AO}{OB} = -\frac{b^2}{a^2}\left(\frac{2b^2c^2 + a^2c^2 + a^2b^2 - a^4}{2a^2c^2 + b^2c^2 + a^2b^2 - b^4}\right).$$

If the cevian ratios for the negative Brocard point are placed in the equations of Theorem 8, the results of Theorem 8B will be obtained. So generally, when determining the radical axis of the circumcircle and pedal circle, ratio values for either of the isogonal conjugates may be used in Theorem 8.

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Operating on (1) with this operator we obtain

$$g(D)f(D)y = 0. \tag{3}$$

The procedure from this point is to solve (3) noting that, generally, by raising the order of the equation, extraneous solutions will have been introduced. Any solution of (3) will be subject to verification in (1). The solution of (3) is readily obtained when both $g(D)$ and $f(D)$ have constant coefficients.

Selection of an operator. We shall demonstrate the selection of an operator when

$$R(x) = \sum_{i=1}^{n} \sum_{j=0}^{n'} a_j x^j e^{\alpha_i x} \qquad (\alpha_i \text{ a complex number})$$

which is the most general function consisting of products of polynomials, sines, cosines, hyperbolic sines, hyperbolic cosines, and exponentials. This will be recognized as the most general right hand member that can be solved using undetermined coefficients.

LEMMA I. *If $D^t = d^t/dx^t$ and P_n is a polynomial of degree n, then $D^t(P_n e^{\alpha x}) = e^{\alpha x}[D+\alpha]^t P_n$ where α and t are constants.*

Proof. Given two functions h_1 and h_2 that are differentiable, then

$$D^t(h_1h_2) = h_2D^th_1 + n(Dh_2)(D^{t-1}h_1) + \frac{n(n-1)}{2!}(D^2h_2)(D^{t-2}h_1) + \cdots + h_1D^th_2$$

which is Leibnitz' Rule [1]. Letting $h_1 = P_n$ and $h_2 = e^{\alpha x}$, we see that

$$D^t(P_ne^{\alpha x}) = e^{\alpha x}\left[\alpha^tP_n + t\alpha^{t-1}DP_n + \frac{t(t-1)}{2!}\alpha^{t-2}D^2P_n + \cdots + D^tP_n\right]$$
$$= e^{\alpha x}[D+\alpha]^tP_n.$$

LEMMA II. *If $D^* = D - \beta$, then $D^{*t}(P_ne^{\alpha x}) = e^{\alpha x}[D-\beta+\alpha]^tP_n$ where β is a constant.*

Proof. $[D-\beta]^t(P_ne^{\alpha x})$

$$= \left[D^t - \beta tD^{t-1} + \frac{t(t-1)}{2!}\beta^2D^{t-2} + \cdots\right]P_ne^{\alpha x}$$

$$= e^{\alpha x}\left[(D+\alpha)^t - t\beta(D+\alpha)^{t-1} + \frac{t(t-1)}{2!}\beta^2(D+\alpha)^{t-2} + \cdots\right]P_n$$

using Lemma I.

Finally, $[D-\beta]^t(P_ne^{\alpha x}) = e^{\alpha x}[D-\beta+\alpha]^tP_n$.

We now use these two results for further calculations. Let

$$G_j(x) = x^je^{\alpha_i x}.$$

We seek an operator $g_i(D)$ such that $g_i(D)G_j(x) = 0$. It is readily verified that

$$(D - \alpha_i)\ G_0(x) = 0$$
$$(D - \alpha_i)^2 G_1(x) = 0$$
$$\cdots\cdots\cdots.$$

Now consider the k-th term. We seek a number p such that

$$(D - \alpha_i)^p G_k(x) = 0.$$

Using Lemma II we obtain

$$(D - \alpha_i)^p (x^k e^{\alpha_i x}) = e^{\alpha_i x} D^p x^k.$$

This clearly vanishes for $p > k$. It follows that

$$g_i(D) = (D - \alpha_i)^m, \qquad m > j.$$

Now consider the same problem from a different approach. Assume

$$g_i(D) = (D - \alpha_i)^m, \qquad m > j$$

and operate on $G_j(x)$. We obtain

$$g_i(D) G_j(x) = (D - \alpha_i)^m x^j e^{\alpha_i x} = 0$$

by Lemma II. We can summarize these results in Lemma III.

LEMMA III. *Given an operator $g_i(D)$ and a function $G_j(x) = x^j e^{\alpha_i x}$; then $g_i(D)G_j(x) = 0$ if and only if $g_i(D) = (D - \alpha_i)^m$, $(m > j)$.*

It follows immediately that

$$g_i(D) \sum_{j=0}^{n'} a_j x^j e^{\alpha_i x} = 0 \qquad \text{for } m > n'.$$

Now define an operator $g_t(D) = (D - \alpha_t)^m$. Then

$$g_t(D) R(x) = g_t(D) \sum_{i=1}^{n} \sum_{j=0}^{n'} a_j x^j e^{\alpha_i x}$$

will reduce to zero each term for which $t = i$. It follows that

$$\prod_{t=1}^{n} g_t(D) R(x) = 0$$

and, finally, that

$$g(D) = \prod_{t=1}^{n} (D - \alpha_t)^m, \qquad m > n'.$$

These results can be summarized in the following

THEOREM. *Given the function*

$$R(x) = \sum_{i=1}^{n} \sum_{j=0}^{n'} a_j x^j e^{\alpha_i x}$$

and the operator

$$g(D) = \prod_{t=1}^{n} (D - \alpha_t)^m \qquad m > n',$$

then $g(D)R(x) = 0$.

Conclusions. The results of this work can best be evaluated by solving a sample problem. Let us seek the solution of the equation:

$$(D^2 + 1)y = xe^x \cos x.$$

Using the notation of this paper, we have

$$R(x) = \tfrac{1}{2}x\left[e^{(1+i)x} + e^{(1-i)x}\right].$$

Choose $m = 2(m > 1)$ and $n = 2$ since there are two values for α. It follows that

$$g(D) = [D - (1 + i)]^2[D - (1 - i)]^2$$

and we solve the equation

$$[D - (1 + i)]^2[D - (1 - i)]^2 y = 0.$$

The solution is easily found to be

$$y = (A + Bx)e^{(1+i)x} + (C + Dx)e^{(1-i)x}$$

or $y = e^x[c_1 \cos x + c_2 \sin x + x(c_3 \cos x + c_4 \sin x)]$. Forcing this to satisfy the given differential equations, we obtain

$$c_1 = \frac{-2}{25}, \qquad c_2 = \frac{-14}{25}, \qquad c_3 = \frac{1}{5}, \qquad c_4 = \frac{2}{5}.$$

This gives a particular solution to the nonhomogeneous equation which must be added to a solution of the homogeneous equation to yield the general solution.

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EXTENDAPAWN—AN INDUCTIVE ANALYSIS

JOHN R. BROWN, Seattle, Washington

Introduction. The goal of this paper is to explain the complete analysis of a game called EXTENDAPAWN, which is the natural extension of the more simple game of Hexapawn (as described by Martin Gardner in the May, 1962 issue of *Scientific American*).

and the operator

$$g(D) = \prod_{t=1}^{n} (D - \alpha_t)^m \qquad m > n',$$

then $g(D)R(x) = 0$.

Conclusions. The results of this work can best be evaluated by solving a sample problem. Let us seek the solution of the equation:

$$(D^2 + 1)y = xe^x \cos x.$$

Using the notation of this paper, we have

$$R(x) = \tfrac{1}{2}x\left[e^{(1+i)x} + e^{(1-i)x}\right].$$

Choose $m = 2(m > 1)$ and $n = 2$ since there are two values for α. It follows that

$$g(D) = [D - (1 + i)]^2[D - (1 - i)]^2$$

and we solve the equation

$$[D - (1 + i)]^2[D - (1 - i)]^2 y = 0.$$

The solution is easily found to be

$$y = (A + Bx)e^{(1+i)x} + (C + Dx)e^{(1-i)x}$$

or $y = e^x[c_1 \cos x + c_2 \sin x + x(c_3 \cos x + c_4 \sin x)]$. Forcing this to satisfy the given differential equations, we obtain

$$c_1 = \frac{-2}{25}, \qquad c_2 = \frac{-14}{25}, \qquad c_3 = \frac{1}{5}, \qquad c_4 = \frac{2}{5}.$$

This gives a particular solution to the nonhomogeneous equation which must be added to a solution of the homogeneous equation to yield the general solution.

References

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Introduction. The goal of this paper is to explain the complete analysis of a game called EXTENDAPAWN, which is the natural extension of the more simple game of Hexapawn (as described by Martin Gardner in the May, 1962 issue of *Scientific American*).

DEFINITION. *EXTENDAPAWN* is played on a rectangular board consisting of 3 rows and n columns together.

The game is begun with pawns of one kind, say X, filling the base row and pawns called O filling the top row, the middle row being empty. (See Fig. 1.)

A move by a player consists of

1. moving a pawn straight forward one square into an empty square, or
2. "capturing" one of the opponent's pawns diagonally. The object of the game of n-columns is to win by:

Type 1. moving into the opponent's row with one pawn, or

Type 2. depriving the opponent of any further move (i.e., by moving last).

The proposed problem is as follows: Assuming the best possible strategy is used by each player, find the winner for each value of n (i.e., a game with n columns). An inductive algorithm is employed which reduces the original "n" game to games with a total of fewer columns than in the original game.

Section I. Playing the Game. An example of this method is easily illustrated for the case $n=5$.

Assume everything necessary is known about all games for $n<5$. The beginning of the game is:

O	O	O	O	O
X	X	X	X	X

FIG. 1.

Let X move the column 1 pawn forward first: i.e.,

O	O	O	O	O
X				
	X	X	X	X

Now clearly O has a choice. His column 2 pawn may either (1) be moved forward out of danger of the aggressive X pawn, or (2) capture that X pawn. If the capture is made, the following situation is created:

O		O	O	O
O				
	X	X	X	X

Notice now that X is left without a choice, since failing to capture the threatening O will result in a Type 1 win for O. Assuming then that X captures as necessary, the game reduces to a game of fewer columns (in this case 3) with O now having the first move.

Notice now that O caused the game to be reduced, as shown, *and* O has the first move in the reduced game; hence O could have either captured or not captured when the choice arose depending upon the already known outcome of the lesser column game (in this case 3 columns) that would remain to be played.

Working with this principle in the particular game chosen, O has a choice of reducing to a 3 column game (with the first move in that game) or not capturing. Given now that the player moving first in a 3 column game cannot force a win, O will not choose to reduce to the 3 column game. Ignoring for now the known fact that moving on the end in a 4 column game produces a win, we see that the game would proceed:

O		O	O	O
X	O			
	X	X	X	X

Notice now that X is given a similar choice, and that if X captures, the game will be reduced to a 2 column game (in which X will have the first move and cannot win). The obvious X move is then forward:

O		O	O	O
X	O	X		
	X		X	X

Now O can capture and reduce to a 1 column game with first move which is obviously a winning strategy. Hence if X moves the column 1 pawn first, O can win the game. This, then, is not a winning first move for X in a 5 column game.

Next try moving the column 2 X pawn straight forward as a first move. Thus:

	O	O	O	O	O
*		X			
	X		X	X	X

Here notice that O must capture the threatening X or lose the game on the next move (Type 1 win for X), but O has a choice of capturing from either side, and it is necessary also to notice that whatever the O move may be, X must capture on the next move from the same side.

Example.

	O	O	O	O
	O			
X		X	X	X

If X does not capture from the left the game will proceed as follows:

	O	O	O	O
	X			
X			X	X

	O		O	O
	X	O		
X			X	X

	O		O	O
	X	X		
X				X

	O			O
	X	O		
X				X

And O wins handily. Hence in situation * above, O has the choice of reducing to a 4 column game in which X will have moved first on the end, or to a 2 column game in which X will have moved first on the end and another untouched 3 column game. O may then appraise the potential of both possibilities. Knowing all about smaller games, O recognizes that X can win a 4 column game by moving first on the end; the remaining possible move for O, however, is a very powerful one. Given the fact that O can emerge from a 2 column game with his choice of move, he need merely know by analyzing the game that remains (in this case a 3 column game) whether it is most advantageous to have the first or second move in that game, and play the 2 column game accordingly. (This same principle obviously holds for any remaining game regardless of the value of n, so it is established that X can never win by moving first the column 2 pawn.) X then must try moving first the column 3 pawn:

O	O	O	O	O
		X		
X	X		X	X

Here (due to symmetry and the fact that X must capture from the same side), the game is reduced to a 3 column game in which X will have moved first on the end and a 1 column game (untouched): i.e.,

O	O	O		O
		X		
X	X			X

For this particular case (5 column game) the outcome is then inevitable, since X can force O to move last in the 3 column game, giving X the first move in the 1 column game and a subsequent over-all win in the original game.

Section II. The Inductive Approach. In general, it may occur that O will be left a choice either to move last or not in one segmented part of the original game depending upon his subsequent desire to move second or first (respectively) in the remainder of the game. The general game of n columns then becomes increasingly complex, and it becomes necessary to know winning possibilities for more general situations.

As previously mentioned for the game of 5 columns, it can be shown that in any given game (after at least one move):

LEMMA 1. *A capture by one player from either the left or right demands a capture by the opposing player from the same direction.*

Proof. Because of symmetry, this proof will concern only a capture from the left side forcing a reply of a capture from the same side. Let us examine the following situation:

O		O		O	O
	O		- -		
X	X	X		X	X

K

O		O		O	O
	X		- -		
	X	X		X	X

K

(Here K represents the number of columns to the right of the column initially moved in by O.)

Three cases must be considered: (1) $K=0$, (2) $K=1$, and (3) $K\geqq 2$.

Case 1. In the case that $K=0$, i.e., there are no columns to the right, then it is impossible for O to capture from the other side, but the following may occur:

	O	O	
- - -			O
	X	X	X

	O	O	
- - -			X
	X		X

	O		
---		O	X
	X		X

	O		X
---		O	
	X		X

and X wins. Here it is apparent that O would have to capture from the left side after X did, or lose immediately as shown.

Case 2. Where there is but one column to the right the outcome is equally immediate in the event that the given principle is not followed: i.e., if $K=1$,

	O	O		O
--			O	
	X	X	X	X

	O	O		O
--			X	
	X		X	X

	O	O		
--			O	
	X		X	X

	O	O		
--			O	X
	X		X	

and X wins.

Case 3. If $K \geqq 2$, the following will occur:

	O		O	O
--		O		
	X	X	X	X

	O		O	O
		X		
		X	X	X

	O			O
		O		
		X	X	X

	O			O
		O	X	
		X		X

	O			
		O	O	
		X		X

	O			
		O	X	
		X		

and X wins. It should be noticed that only the next two columns are needed by X to produce a win; hence this result applies to any configuration for $K \geqq 2$. This completes the proof of Lemma 1.

With the establishment of this all-important principle, the reader may easily verify now that every move by either player (whether it be a capture or a move straight forward) forces reduction to a reduced game or games of fewer than the original number of columns. Thus, it becomes clear that knowledge of smaller games and a means of combining the potentials of those smaller games is necessary in order to play a particular n column game with the best possible strategy.

Now that the reader has become somewhat familiar with the playing of the game, let us launch directly into a more concise treatment of Extendapawn with a few more definitions and the fundamental theorem.

It will be shown through use of a proof by induction just how a player can win any given game in which that player has the advantage. In general, given any winning situation, it will be shown how to move so that after 1, 2, or 3 moves said player will be in a winning situation with less columns.

Section III. Definitions.

Current Game. A game with at least two columns in which a single move has been made on the end. Example:

	O	O	
O			- - -
X	X	X	

Inductive Situation. At most one current game and/or any number of untouched games.

Parity. Untouched games of n columns where $n \equiv 1, 4, 5, 7$, or $8 \pmod{10}$ have parity 1; all other untouched games have parity 0. A current game of $n \pmod{10}$ columns has the parity of an untouched game of $n-2 \pmod{10}$ columns.

Parity of Inductive Situation. The parity of an inductive situation is the sum (mod 2) of the parities of the games.

Advantage. Given any inductive situation, if the parity of the inductive situation is 1 or the current game has $n \equiv 0$ or $2 \pmod{5}$ columns, then the player with the next move (referred to as the first player) is by definition said to have the "advantage."

LEMMA 2. *Any inductive situation can and must be reduced to another inductive situation (consisting of fewer untouched total columns) after at most 3 moves.*

Proof. Denote by E any inductive situation which contains a current game, and by N any inductive situation containing only untouched games.

Case 1. Given that an inductive situation E exists it is obvious that the first move in that inductive situation is strictly limited to the current game involved. Example:

	O	O	O	
O				- -
X	X	X	X	

In fact, the only choice (as diagrammed above) is that of push or capture with the X-pawn in the second column.

1. If the first player pushes, then an inductive situation of one fewer untouched columns than in the original results, for it is easy to see that column 1 will never be further disturbed.

	O	O	O	
O	X			- - -
X		X	X	

2. If the first player captures with the column 2 pawn, then (by Lemma 1) the second player is forced to recapture with his column 2 pawn, thus resulting in

		O	O	
O				- - -
X		X	X	

which is not a current game; hence the result after 2 moves is an inductive situation N of one fewer untouched columns than the original.

So it is seen that in the E situation either 1 or 2 moves must produce a new inductive situation with fewer untouched columns than the original.

Case 2. Now assume that an inductive situation N (i.e., containing no current game) exists. Three separate actions by the first player must be investigated:

1. If there is a 1 column game contained in the inductive situation N and the first move is in that 1 column game, then obviously a new inductive situation N is immediately produced having 1 less untouched column than the original. Note that only 1 move was required.

2. If the first move is on the end of any of the untouched games comprising the inductive situation N, then that particular game (by definition) becomes a current game; hence with only 1 move an inductive situation E is produced. The number of untouched columns here was reduced by one.

3. If neither (1) nor (2) occurs, then the action in general can be described by the figures below:

	O	O	O	O	
					- -
	X	X	X	X	

FIG. 1.

	O	O	O	O	
- -			X		- -
	X	X		X	

FIG. 2.

Obviously O must capture, and Lemma 1 stipulates that X follows with a capture from the same side.

	O		O	O	
			O		- -
	X	X		X	

FIG. 3.

	O		O	O	
- -			X		- -
	X			X	

FIG. 4.

Notice that the original untouched game has now been segmented into a current game and another untouched game; 3 moves were taken in producing the resulting E inductive situation, while the total number of untouched columns in the new configuration is two less than in the original inductive situation N. Therefore, as a result of this lemma, we are able to reduce any inductive situation to a "smaller" inductive situation with at most 3 moves.

Briefly then, the strategy will be as follows: Denote the parity of an inductive situation by P, and the parity of the resulting inductive situation (i.e., after the required 1, 2, or 3 moves) by P'. Given any inductive situation (E or N), then the first player would like to force a new inductive situation of fewer columns where $P'=0$ if 1 or 3 moves are required or $P'=1$ if 2 moves are required, thus either retaining or gaining the advantage. Obviously the second player must attempt to prevent the first player from accomplishing this task.

Section IV. The Theorem.

THEOREM. *The player with the advantage can always force a win.*

Proof. Four separate cases will be considered in the proof in order to exhaust all possibilities.

Case 1. The given situation contains a current game of n columns where $n\equiv 0$ or 2 (mod 5). All possible activity in this situation can be easily summarized with the following table:

n (mod 10)	$n-2$ (mod 10)	par$(n-2)$	n' (mod 10) result of push	$n'-2$ (mod 10)	par$(n'-2)$
0	8	1	9	7	1
2	0	0	1	9	0
5	3	0	4	2	0
7	5	1	6	4	1
A	B	C	D	E	F

n^* (mod 10) result of capture	par(n^*)
8	1
0	0
3	0
5	1
G	H

Column A indicates the number of columns (mod 10) of the current game. Since by definition the parity of a current game is the parity of an untouched game of 2 fewer columns (mod 10), column B above gives the number of columns of the corresponding untouched game, and column C contains then the parity of the current game in question. Given that the first player is confronted with a situation containing a current game of n columns as indicated above, column D gives the number of columns of a new current game created as a result of a push in that situation. Columns E and F then function exactly as did columns B and C for the original current game. On the other hand, column G gives the number of columns of the resulting untouched game should the first player choose to capture, and column H contains the parity of that untouched game.

From this table then it can be seen that neither a push nor a capture causes a change in over-all parity, so that the correct strategy for Case 1 is:

1. to capture if $P=1$ (2 move induction) or
2. to push if $P=0$ (1 move induction)

where both (1) and (2) retain the advantage (see definition). Note that the lower half of the table is directly related to the upper half by the following property:

$$\text{par}(n) \equiv [\text{par}(n+5)+1] \pmod 2 \qquad \text{(see definition of parity)}$$

For example: $\text{par}(1)=1 \qquad \text{par}(6)=0$

$$1 = \text{par}(1) \equiv [\text{par}(6)+1] \pmod 2 \equiv [0+1] \pmod 2 = 1.$$

Through use of this property of symmetry a great deal of redundancy can be avoided in formation of the tables summarizing Cases 2–4. In the remaining cases only the upper portion of the tables will be given.

Case 2. The given situation contains a current game of n columns where $n \not\equiv 0$ or 2 (mod 5).

n(mod 10)	$n-2$	par($n-2$)	n'(mod 10) result of push	$n'-2$	par($n'-2$)	n^*(mod 10) result of capture	par(n^*)
1	9	0	0	8	1	9	0
3	1	1	2	0	0	1	1
4	2	0	3	1	1	2	0
		A			B		C

In this table column B represents results of a 1-move induction, while column C represents results of a 2-move induction. Upon comparison of corresponding entries in columns A and B, and also for A and C, it becomes clear that, regardless of the move in this situation, the advantage will remain unchanged.

Case 3. Let us now determine the potential of an end move in an n-column game for a situation which contains no current game.

1. If $n \equiv 0$ or 2 (mod 5), the end move is never a winning one (see Case 1). It may be interesting to note, however, that if the first player is already losing the game, this may be a good move, for it forces the second player to compute and respond with caution.

2. If $n \not\equiv 0$ or 2 (mod 5), then the following table applies:

n (mod 10)	par(n)	$n-2$ (mod 10)	par($n-2$)
1	1	9	0
3	0	1	1
4	1	2	0
A	B	C	D

Here column A gives the number of columns (mod 10) of an untouched game in which the first player has considered moving. Column B gives the parity of that game. Column A also gives the number of columns of the current game resulting from the end move by the first player. Column C contains the number of columns of the corresponding untouched game from which the parity of the current game is obtained. That parity, then, is given in column D. Notice then that if the end move is made, the parity changes as well as the roles of first and second player; therefore the advantage remains unchanged.

Case 4. Assume there exists no current game in the given situation, and the first player moves in column r from one end and r' from the other end of an n-column game. Therefore, $r+r'=n+1$; r, $r'>1$ and, by Lemma 2, a 3-move induction is involved. It is clear that a capture must follow the first move and, by Lemma 1, yet another capture from the same side must follow. Three moves in all are necessary to produce another inductive situation of fewer untouched columns than the original. It can be easily verified that the n-column game will have been divided into

1. current game of r columns + untouched game of $r'-2$ columns, or
2. current game of r' columns + untouched game of $r-2$ columns.

In either case, again by the definition of the parity of a current game, the sum of the parities of the two games resulting from the 3-move induction is given by $[\text{par}(r-2)+\text{par}(r'-2)]$ (mod 2).

1. If both $r \not\equiv 0$ or 2 (mod 5) and $r' \not\equiv 0$ or 2 (mod 5) then it is shown in Table 4a that the advantage is not changed by this move since there is a change in over-all parity from one inductive situation to the next but (from the fact that a 3-move induction is involved) also a change of move (i.e., the roles of first and second player have changed). The fact that the over-all parity has changed stems from the result that the sum of the parities of the two smaller games is different from that of the original n-column game; i.e., $S(\text{mod } 2) \equiv \text{par}(r-2) + \text{par}(r'-2) + \text{par}(n) \equiv 1 (\text{mod } 2)$.

Note. Here it should be observed that due to the relationships,

$$\text{par}(r) \equiv [\text{par}(r+5)+1] \pmod 2$$
$$\text{par}(r') \equiv [\text{par}(r'+5)+1] \pmod 2$$

and
$$\text{par}(n) \equiv [\text{par}(n+5)+1] \pmod 2$$

it is possible to delete 3/4 of the table which would result (see the remarks concluding Case 1), and Table 4a below can be seen to describe fully the courses of action which might result.

r(mod 10)	par($r-2$)	r'(mod 10)	par($r'-2$)	n(mod 10)	par(n)	S(mod 2)
1	0	1	0	1	1	1
		3	1	3	0	1
		4	0	4	1	1
3	1	3	1	5	1	1
		4	0	6	0	1
4	0	4	0	7	1	1

TABLE 4a.

2. If $r \equiv 0$ or 2(mod 5), then the second player can surely gain the advantage by capturing with the pawn from column $r+1$ and application of the principles derived in Case 1. If $r' \equiv 0$ or 2(mod 5) also, then the second player may capture from either side with equally satisfactory results. If, however, $r' \not\equiv 0$ or 2(mod 5) we must again calculate $\text{par}(n)+\text{par}(r-2)+\text{par}(r'-2) \equiv S(\text{mod } 2)$.

r(mod 10)	par($r-2$)	r'(mod 10)	par($r'-2$)	n(mod 10)	par(n)	S(mod 2)
0	1	1	0	0	0	1
		3	1	2	0	0
		4	0	3	0	1
2	0	1	0	2	0	0
		3	1	4	1	0
		4	0	5	1	1

TABLE 4b.

From this table, then, the value of a capture by the second player with the pawn in column $r-1$ can be obtained. S(mod 2) denotes the change in parity from the original situation to the new inductive situation. If $S\equiv 1 \pmod 2$, a change in parity has occurred; if $S\equiv 0 \pmod 2$, no change has occurred. Thus, if in the original situation $P=0$ and a particular response (as indicated by some row of the table) results in $S\equiv 0 \pmod 2$, then this response is obviously less attractive than a capture from the other side [with which the advantage could have been retained (Case 1)]. It should be noticed, however, that if the second player has the advantage in a given inductive situation, and if the first player can move in a manner forcing the second player to calculate carefully or *lose* the advantage in the next inductive situation, then such a move is a good move for the first player. (*Note.* Move in $r\equiv 0$ or $2 \pmod 5$ is always bad for first player if $P=1$.) (See Case 1.)

The cases for ($P=0$, $S\equiv 1$), ($P=1$, $S\equiv 0$), and ($P=1$, $S\equiv 1$) are as easily discussed as the preceding, but will simply be summarized here. To illustrate then the value of the move in column $r\equiv 0, 2 \pmod 5$ by the first player and the corresponding value of a capture by second player with the pawn in column $r-1$ for all possible circumstances, the following table is constructed:

P	S(mod 2)	Value to first player	Value to second player
0	0	Good	Bad
	1	Bad	Good
1	0	Bad	Bad
	1	Bad	Good

TABLE 4c.

Tables 4a, 4b, 4c, then completely summarize all situations encountered in Case

4, illustrating that (aside from careless moves) the player with the advantage can always force a win.

Section V. Conclusion. This completes the proof of the theorem. In particular, the theorem has established that the first player can win any single game of n columns where $n \equiv 1, 4, 5, 7$, or $8 \pmod{10}$. Otherwise the second player can produce a win. It should be noticed that more than the original problem as stated has been solved. The solution has been extended to any combination of games (i.e., any inductive situation).

In addition we have completely analyzed every sequence of moves between any two inductive situations, providing a complete collection of all moves which retain the advantage. All other moves lose the advantage. Any move by the opponent which is not covered in the proof of the theorem will result in an immediate (Type 1) win for the player.

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The results given here form a part of the author's master's dissertation (University of Washington, 1964) written under the direction of Professor John L. Selfridge.

AN INTERESTING THEOREM?

If $y=f(x)$ is C^2 on $[a, b]$ and $f'(x)>0$ on (a, b), then $f''(x)=0$ on (a, b).

Proof.

$$\frac{d}{dy}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dydx} = \frac{d}{dx}\left(\frac{dy}{dy}\right) = \frac{d}{dx}(1) = 0.$$

But

$$\frac{d}{dy}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}\left(\frac{dx}{dy}\right)$$

and hence

$$\frac{d^2y}{dx^2} = 0.$$

VECTOR SOLUTIONS OF GEOMETRIC PROBLEMS AND THEIR GENERALIZATIONS

A. R. AMIR-MOÉZ, Texas Technological College

This note intends to point out some advantages of vector approach to problems of Euclidean geometry. We shall solve a sample problem and generalize the proposition to an n-dimensional real unitary (Euclidean) space. This sample problem suggests the possibility of other problems which can be assigned as exercises to students in an elementary course in linear spaces.

1. *Notations.* We shall use capital letters both for vectors and points. For two vectors A and B the inner product of A and B is denoted by (A, B). The norm of a vector A is denoted by $|A|$. A line segment between the points A and B will be denoted by AB.

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4, illustrating that (aside from careless moves) the player with the advantage can always force a win.

Section V. Conclusion. This completes the proof of the theorem. In particular, the theorem has established that the first player can win any single game of n columns where $n \equiv 1$, 4, 5, 7, or 8 (mod 10). Otherwise the second player can produce a win. It should be noticed that more than the original problem as stated has been solved. The solution has been extended to any combination of games (i.e., any inductive situation).

In addition we have completely analyzed every sequence of moves between any two inductive situations, providing a complete collection of all moves which retain the advantage. All other moves lose the advantage. Any move by the opponent which is not covered in the proof of the theorem will result in an immediate (Type 1) win for the player.

The author acknowledges with appreciation the numerous suggestions of Professor John L. Selfridge and Dr. C. Y. Lee.

The results given here form a part of the author's master's dissertation (University of Washington, 1964) written under the direction of Professor John L. Selfridge.

AN INTERESTING THEOREM?

If $y=f(x)$ is C^2 on $[a, b]$ and $f'(x)>0$ on (a, b), then $f''(x)=0$ on (a, b).

Proof.
$$\frac{d}{dy}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dydx} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(1) = 0.$$

But
$$\frac{d}{dy}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}\left(\frac{dx}{dy}\right)$$

and hence
$$\frac{d^2y}{dx^2} = 0.$$

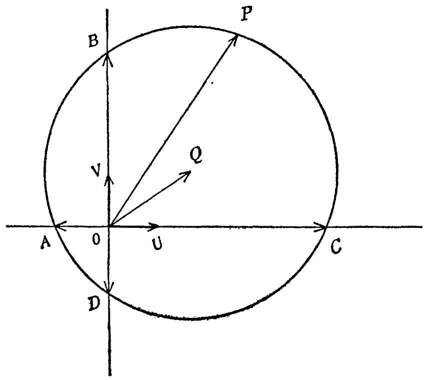
VECTOR SOLUTIONS OF GEOMETRIC PROBLEMS AND THEIR GENERALIZATIONS

A. R. AMIR-MOÉZ, Texas Technological College

This note intends to point out some advantages of vector approach to problems of Euclidean geometry. We shall solve a sample problem and generalize the proposition to an n-dimensional real unitary (Euclidean) space. This sample problem suggests the possibility of other problems which can be assigned as exercises to students in an elementary course in linear spaces.

1. *Notations.* We shall use capital letters both for vectors and points. For two vectors A and B the inner product of A and B is denoted by (A, B). The norm of a vector A is denoted by $|A|$. A line segment between the points A and B will be denoted by AB.

2. THEOREM. *Let* (Q) *be a circle of radius* r *and center at* Q. *Consider a point* O *inside* (Q). *Let* AC *and* BD *be any two chords of* (Q) *through* O *which are perpendicular to each other. Then* $(OA)^2+(OB)^2+(OC)^2+(OD)^2=4r^2$, *i.e., this sum is independent of the position of* O *and the lines* AC *and* BD.



Proof. First we shall translate the theorem into the language of vectors. Let us choose O for the zero vector. Let U and V be two unit vectors respectively on C and B (Fig. 1). Then $P=tU$, $P=sV$, respectively, will be vector equations of lines AC and BD. Let P be a point on the circle. Then $|P-Q|=r$ is the vector equation of (Q). In all equations P is a variable vector. Now the vectors C and D are obtained from

$$|P-O|=r \quad \text{and} \quad P=tU.$$

Substituting tU for P in the first equation and considering properties of the inner product we get

$$t^2-2t(Q,\,U)+|Q|^2-r^2=0. \tag{1}$$

This equation has two roots t_1 and t_2 such that $A=t_1U$ and $C=t_2U$. Similarly for B and D we get $B=s_1V$ and $D=s_2V$, where s_1 and s_2 are roots of

$$s^2-2s(Q,\,V)+|Q|^2-r^2=0. \tag{2}$$

Now we see that

$$(OA)^2+(OB)^2+(OC)^2+(OD)^2=t_1^2+t_2^2+s_1^2+s_2^2.$$

One may compute the roots of (1) and (2) and substitute in $t_1^2+t_2^2+s_1^2+s_2^2$, keeping in mind that $(U,\,V)=0$, and obtain $4r^2$ for it. But we shall make some observations to shorten the arithmetic of the problem. We observe that $(Q,\,U)$ and $(Q,\,V)$ are components of Q on U and V. Thus

$$(Q,\,U)^2+(Q,\,V)^2=|Q|^2.$$

On the other hand

$$t_1 + t_2 = 2(Q, U), \qquad s_1 + s_2 = 2(Q, V), \qquad t_1t_2 = |Q|^2 - r^2 = s_1s_2.$$

Therefore we write

$$\begin{aligned} t_1^2 + t_2^2 + s_1^2 + s_2^2 &= (t_1 + t_2)^2 + (s_1 + s_2)^2 - 2t_1t_2 - 2s_1s_2 \\ &= 4[(Q, U)^2 + (Q, V)^2] - 4[|Q|^2 - r^2] = 4r^2. \end{aligned}$$

3. *A generalization.* Here we state the hypotheses of the theorem. We proceed as in Section 2. Then we give the conclusion when we obtain it.

Let P be on the $(n-1)$-sphere of center Q and radius r, in a real Euclidean space. Then $|P-Q| = r$. Let $\{U_1, \cdots, U_n\}$ be an orthonormal set of vectors. Then a vector on U_i is of the form $P = t_iU_i$. The common points of the sphere and the line are obtained from

$$t_i^2 - 2t_i(Q, U_i) + |Q|^2 - r^2 = 0. \tag{3}$$

This equation has two roots t_{i1} and t_{i2}. We shall compute $\sum_{i=1}^n (t_{i1}^2 + t_{i2}^2)$. Again we observe that

$$t_{i1} + t_{i2} = 2(Q, U_i), \qquad t_{i1}t_{i2} = |Q|^2 - r^2, \quad \text{for all } i.$$

On the other hand

$$\sum_{i=1}^n (Q, U_i)^2 = |Q|^2.$$

Thus we have

$$\sum_{i=1}^n (t_{i1}^2 + t_{i2}^2) = \sum_{i=1}^n (t_{i1} + t_{i2})^2 - 2\sum_{i=1}^n t_{i1}t_{i2} = 4\sum_{i=1}^n (Q, U_i)^2 - 2n[|Q|^2 - r^2].$$

Thus the conclusion is

$$\sum_{i=1}^n (t_{i1}^2 + t_{i2}^2) = 2nr^2 - (2n - 4)|Q|^2,$$

where the right hand side of the equation is a constant.

4. COROLLARY. *In section 3 if $r = |Q|$, then one of the roots of (3), say t_{i2}, is zero. Thus $\sum_{i=1}^n t_{i1}^2 = 4r^2$.*

Let t_{i1} be called t_i. Then we can prove that $\sum_{i=1}^n t_iU_i = 2Q$, which is a generalization of the fact that a right triangle can be inscribed in a half circle. But we are looking at the proposition differently, i.e., twice the median corresponding to the hypotenuse is the diameter of the circumscribed circle.

Generalization of the preceding ideas to complex spaces is more interesting. We leave it to the reader.

A FACTORIAL CONJECTURE

MYRON TEPPER, University of Illinois

In this paper we are motivated to claim the identity

$$r! = \sum_{i=0}^{r} (-1)^i \binom{r}{i} (n-i)^r$$

for every positive integer r and for any n from a consideration of certain numerical data:

I.

n	n^1	T_1
1	1	
		1
2	2	
		1
3	3	
		1
4	4	
		1
5	5	
.	.	
.	.	
.	.	

II.

n	n^2	T_1	T_2
1	1		
		3	
2	4		2
		5	
3	9		2
		7	
4	16		2
		9	
5	25		
.	.		
.	.		
.	.		

where T_x = the tabular difference of the preceding column.

III.

n	n^3	T_1	T_2	T_3
1	1			
		7		
2	8		12	
		19		6
3	27		18	
		37		6
4	64		24	
		61		
5	125			
.	.			
.	.			
.	.			

IV.

n	n^4	T_1	T_2	T_3	T_4
1	1				
		15			
2	16		50		
		65		60	
3	81		110		24
		175		84	
4	256		194		24
		369		108	
5	625		302		
		671			
6	1296				
.	.				
.	.				
.	.				

In general terms the preceding arrays can be summarized as follows:

I. $n-(n-1)=1.$

II. $[n^2-(n-1)^2]-[(n-1)^2-(n-2)^2]$
$=n^2-2(n-1)^2+(n-2)^2=2.$

III. $\{[n^3-(n-1)^3]-[(n-1)^3-(n-2)^3]\}$
$-\{[(n-1)^3-(n-2)^3]-[(n-2)^3-(n-3)^3]\}$
$=n^3-3(n-1)^3+3(n-2)^3-(n-3)^3=6.$

IV. $\{[n^4-(n-1)^4]-[(n-1)^4-(n-2)^4]\}$
$-\{[(n-1)^4-(n-2)^4]-[(n-2)^4-(n-3)^4]\}$
$-\{[(n-1)^4-(n-2)^4]-[(n-2)^4-(n-3)^4]\}$
$+\{[(n-2)^4-(n-3)^4]-[(n-3)^4-(n-4)^4]\}$
$=n^4-4(n-1)^4+6(n-2)^4-4(n-3)^4+(n-4)^4=24.$

In each case the last equality can be verified by simply expanding each of the binomials involved. These considerations clearly suggest that

$$r! = \sum_{i=0}^{r} (-1)^i \binom{r}{i} (n-i)^r$$

for every positive integer r and for any n as stated above.

PROOF OF TEPPER'S FACTORIAL CONJECTURE

CALVIN T. LONG, Washington State University

1. Introduction. In the preceding paper [1], Myron Tepper has stated the following conjecture: *If x is fixed and r is a positive integer, then*

$$r! = \sum_{k=0}^{r} (-1)^k \binom{r}{k} (x-k)^r.$$

In this paper we prove that Tepper's conjecture is correct.

2. *Proof of the theorem.* Since r is a positive integer, if we agree to set $0^0 = 1$ we may write

$$(1)\qquad \begin{aligned} \sum_{k=0}^{r} (-1)^k \binom{r}{k} (x-k)^r &= \sum_{k=0}^{r} (-1)^k \binom{r}{k} \sum_{j=0}^{r} (-1)^j \binom{r}{j} x^{r-j} k^j \\ &= \sum_{j=0}^{r} (-1)^j \binom{r}{j} x^{r-j} \sum_{k=0}^{r} (-1)^k \binom{r}{k} k^j \end{aligned}$$

and the proof depends on evaluating the sums

$$(2)\qquad \sum_{k=0}^{r} (-1)^k \binom{r}{k} k^j$$

for each value of j. The sum (2) is just the sum

$$(3)\qquad \sum_{k=0}^{r} (-1)^k \binom{r}{k} k^j e^{ky}$$

evaluated at $y=0$, and (3) is easily seen to be equal to

$$\begin{aligned} \frac{d^j}{dy^j} \sum_{k=0}^{r} (-1)^k \binom{r}{k} e^{ky} &= (-1)^r \frac{d^j}{dy^j} \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} e^{ky} \\ &= (-1)^r \frac{d^j}{dy^j} (e^y - 1)^r. \end{aligned}$$

In each case the last equality can be verified by simply expanding each of the binomials involved. These considerations clearly suggest that

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For $j=0, 1, \cdots, r-1$, the expression

$$\frac{d^j}{dy^j}(e^y-1)^r$$

is a sum of terms each of which contains a factor e^y-1. For $j=r$,

$$\frac{d^j}{dy^j}(e^y-1)^r$$

contains the term $r!e^{ry}$ and all other terms have a factor e^y-1. Therefore, setting $y=0$ we have that

$$\begin{aligned}\sum_{k=0}^{r}(-1)^k\binom{r}{k}k^j &= 0 \qquad \text{for } j=0,1,\cdots,r-1\\ &= (-1)^r r! \quad \text{for } j=r.\end{aligned}$$

Combining this with (1), we obtain

$$\sum_{k=0}^{r}(-1)^k\binom{r}{k}(x-k)^r = r!$$

as claimed.

Reference

1. M. Tepper, A factorial conjecture, this MAGAZINE, 37 (1965) 303.

THE DIOPHANTINE EQUATION $X^3+Y^3=9Z^3$

J. A. H. HUNTER, Toronto, Canada

Problem No. 20 in Henry E. Dudeney's famous *Canterbury Puzzles* depends on finding positive integral solutions for the above equation. In his solution, Dudeney commented:

"Given a known case for expression of a number as the sum or difference of two cubes we can, by formula, derive from it an infinite number of other cases alternatively positive and negative. So Fermat, starting from the known case $1^3+2^3=9$ *(which we call a "fundamental"), obtained first a negative solution in greater numbers, and from this his positive solution in still larger numbers.*

But there is an infinite number of fundamentals, and I found by trial a negative fundamental in smaller numbers than Fermat's derived negative solution, from which I obtained the result in this problem."

For $j=0, 1, \cdots, r-1$, the expression

$$\frac{d^j}{dy^j}(e^y-1)^r$$

is a sum of terms each of which contains a factor e^y-1. For $j=r$,

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$$\begin{aligned}\sum_{k=0}^{r}(-1)^k\binom{r}{k}k^j &= 0 \qquad \text{for } j = 0, 1, \cdots, r-1\\ &= (-1)^r r! \quad \text{for } j = r.\end{aligned}$$

Combining this with (1), we obtain

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But there is an infinite number of fundamentals, and I found by trial a negative fundamental in smaller numbers than Fermat's derived negative solution, from which I obtained the result in this problem."

Dudeney was referring, of course, to the well-known solution $X=x(x^3+2y^3)$, $Y=-y(y^3+2x^3)$, $Z=z(x^3-y^3)$, where (x, y, z) is any integral solution for $X^3+Y^3=eZ^3$. For the case in question, with $e=9$, Fermat derived the successive integral solutions:

$X=$	2	20	1,88479	12,43617,73399,00948,36481
$Y=$	1	-17	-36520	4,87267,17171,43523,36560
$Z=$	1	7	90391	6,09623,83567,61372,97449

Note that Dudeney used the phrase "*I found by trial.*" That was nearly 60 years ago, long before computers and other modern aids, and at that time the discovery of his special fundamental "by trial" would be a real achievement. It seemed strange to me, as it probably has to many others, that Dudeney never identified his fundamental and never disclosed his trial method. He was not normally so diffident about his considerable mathematical accomplishments; hence his reticence here was all the more surprising.

Now I have identified that mysterious fundamental, and have indeed done so by two quite different theoretical methods—by no means "by trial." These will be outlined separately.

A. Setting $X=ak+2$, $Y=bk+1$, $Z=ck+1$, we have:

$$(a^3 + b^3 - 9c^3)k^2 + 3(2a^2 + b^2 - 9c^2)k = 3(9c - 4a - b).$$

Now, from Fermat's solutions already listed, we have $20^3-17^3=9.7^3$, so $(a^3+b^3-9c^3)$ vanishes with $a=20$, $b=-17$, $c=7$. With these values, however, $(9c-4a-b)$ also vanishes, and we derive $k=0$, leading back to the trivial $X=2$, $Y=1$, $Z=1$ fundamental. But $a^3+b^3-9c^3$ vanishes also with $a=-17$, $b=20$, $c=7$, and this gives $k=37/179$, leading with interchange of X and Y to $X=919$, $Y=-271$, $Z=438$; this must have been Dudeney's special fundamental. Thence, using the Fermat formula, we derive immediately this all-positive integral solution that is so much smaller than Fermat's. This is in fact the solution found by Dudeney:

$$X = 67{,}67024{,}67503 \qquad Y = 41{,}52805{,}64497 \qquad Z = 34{,}86716{,}82660.$$

B. This method involves use of the identity which was stated by E. Lucas in his proof that $X^3+Y^3=eZ^3$ can have integral solutions only if the coefficient e is of the form $mn(m+n)/s^3$. This appeared in 1879 in the *American Journal of Mathematics, Volume* 2; presumably it was unknown to Dudeney when he produced his problem and solution more than 20 years later. Lucas did not apply the identity to the particular case of $e=9$, although in his paper he did list Fermat's results for this case. The identity is:

$$(x^3-y^3+6x^2y+3xy^2)^3+(y^3-x^3+3x^2y+6xy^2)^3=xy(x+y)(3x^2+3xy+3y^2)^3$$

Now, $-9\cdot 1(-9+1)=2^3\cdot 9$. So, in that identity we substitute $x=-9$, $y=1$, whence $(-271)^3+(919)^3=9\cdot 438^3$. This implies $X=919$, $Y=-271$, $Z=438$, which is Dudeney's special fundamental again.

DERIVATIVES OF DETERMINANTS AND OTHER MULTILINEAR FUNCTIONS

KENNETH O. MAY, Carleton College and University of California, Berkeley

From Jacobi's first derivation in 1841 [1] to the most recent rediscovery [2], it has been customary to obtain formulas for the derivatives of a determinant by making full use of the definition to expand, differentiate, and collect. Actually the resulting formulas depend only on the multilinearity of the determinant as a function of its rows, and it is easier to derive general formulas applying to all multilinear functions than it is to derive the special cases for any particular instance such as the dot product, cross product, permanent, or determinant.

To this end let V_i $(i = 1, \cdots, n)$ be finite dimensional vector spaces $(\dim V_i = n_i)$ over a subfield F of the complex numbers, and let a_{ij} be the jth coordinate of $a_i \in V_i$ with respect to the basis $(e_{i1}, \cdots, e_{in_i})$. Let L be a multilinear function from the cartesian product of the V_i to a vector space V over F, and write $L(a) = L(a_1, \cdots, a_n)$. Supposing that the a_i are differentiable functions of a parameter t, we are interested in the derivative of $L(a)$ with respect to t. First we find

$$\frac{\partial L(a)}{\partial a_{ij}} = L(a_1, \cdots, a_{i-1}, e_{ij}, a_{i+1}, \cdots, a_n) \tag{1}$$

by direct calculation. Indeed, since L is linear with respect to each argument,

$$\begin{aligned} L(\cdots, a_{i1}e_{i1} + \cdots + (a_{ij} + \Delta a_{ij})e_{ij} + \cdots + a_{in_i}e_{in_i}, \cdots) \\ - L(\cdots, a_{i1}e_{i1} + \cdots + a_{ij}e_{ij} + \cdots + a_{in_i}e_{in_i}, \cdots) \\ = L(\cdots, \Delta a_{ij}e_{ij}, \cdots). \end{aligned}$$

For determinants, the right member of (1) is the cofactor of a_{ij}, and the equation is the one Jacobi presented in 1841. Writing A_{ij} for the argument in the right side of (1), we have the more familiar form

$$\frac{\partial L(a)}{\partial a_{ij}} = L(A_{ij}). \tag{2}$$

Applying the chain rule to $L(a)$ as a function of the a_{ij}, we find

$$L'(a) = \sum_i \sum_j L(A_{ij})a'_{ij}, \tag{3}$$

which was given for determinants by Jacobi in [1]. To obtain the simpler well-known form we need

$$L(a) = \sum_j L(A_{ij})a_{ij}, \tag{4}$$

which for determinants is the Laplace expansion by elements of the ith row and is easily derived from the multilinearity. Now the inner summation in (3) is seen to be $L(a)$ with the ith argument (the ith "row") replaced by its derivative, and we have Cremona's formula of 1856 [3],

$$L'(a) = \left(\sum_i D_i \right) L(a), \tag{5}$$

where D_i is the operation of "differentiating the ith row," i.e., it replaces $L(a)$ by $L(a_1, \cdots, a_i', \cdots, a_n)$. Iteration yields Teixeira's result of 1880 [4],

$$\frac{d^m L(a)}{dt^m} = \left(\sum_i D_i \right)^m L(a), \tag{6}$$

where the power on the right may be developed by the multinomial expansion.

Obvious specializations yield Leibniz' rule for derivatives of a product and other familiar formulas.

References

1. C. G. J. Jacobi, De formatione et proprietatibus Determinantium, J. Reine Angew. Math., 22 (1841) 285–318. Also in his Werke, 3, 355–392 and separately in German translation, Leipzig, 1896.

2. See the many references under "differentiation" and "differential coefficient" in the index to Sir Thomas Muir, Contributions to the History of Determinants, London, 1930. The most recent example is J. G. Christiano and James E. Hall, this MAGAZINE, 37 (1964) 215–217, which duplicates the complicated derivation by E. Brand in L'Enseignment Math., 6 (1904) 457–459.

3. L. Cremona, Intorno ad un teorema di Abel, Annali di sci. mat. e fis., 7 (1856) 99–105.

4. F. G. Teixeira, Note sur la dérivation des déterminants, Proc. London Math. Soc., 12 (1880) 14 and 212.

THE PEAUCELLIER LINKAGE ON THE SURFACE OF A SPHERE

MICHAEL GOLDBERG, Washington, D. C.

1. Inverse points. A pair of points A and C, which are collinear with the center O of a circle of radius k, are said to be inverse points of the circle if $OA \cdot OC = k^2$. This definition of inverse points is not applicable to points on the surface of a sphere. Instead, the following general definition for inverse points on a plane or a sphere is used [1, 2]:

DEFINITION. *Two points, so situated that every circle or straight line which passes through one of them and cuts a given circle at right angles must also pass through the other, are said to be mutually inverse with respect to the circle.*

Therefore, inverse points in a plane, which are transformed to points on a sphere by stereographic projection, will give inverse points on the sphere since stereographic projection is conformal (angles are preserved).

is seen to be $L(a)$ with the ith argument (the ith "row") replaced by its derivative, and we have Cremona's formula of 1856 [**3**],

$$L'(a) = \left(\sum_i D_i \right) L(a), \tag{5}$$

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MICHAEL GOLDBERG, Washington, D. C.

1. Inverse points. A pair of points A and C, which are collinear with the center O of a circle of radius k, are said to be inverse points of the circle if $OA \cdot OC = k^2$. This definition of inverse points is not applicable to points on the surface of a sphere. Instead, the following general definition for inverse points on a plane or a sphere is used [**1, 2**]:

DEFINITION. *Two points, so situated that every circle or straight line which passes through one of them and cuts a given circle at right angles must also pass through the other, are said to be mutually inverse with respect to the circle.*

Therefore, inverse points in a plane, which are transformed to points on a sphere by stereographic projection, will give inverse points on the sphere since stereographic projection is conformal (angles are preserved).

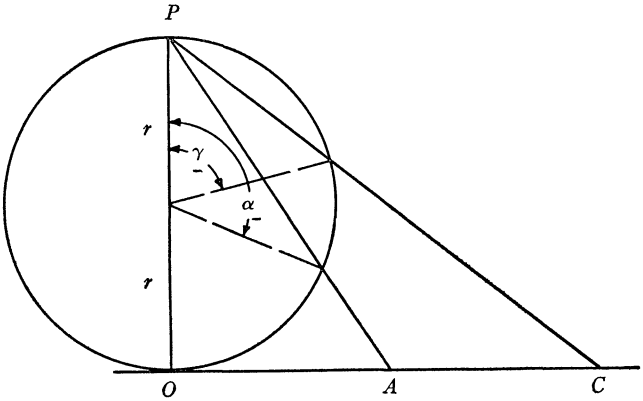

FIG. 1. Inverse points on the plane and on the sphere.

In Figure 1, if A and C are inverse points with respect to a circle lying in a plane which is tangent to the stereographic sphere, then $OA \cdot OC = k^2$. Let the points on the sphere corresponding to A and C have polar distances α and γ. Then, if $2r=1$,

$$OA = \tan(\pi/2 - \alpha/2) = \cot \alpha/2, \qquad OC = \tan(\pi/2 - \gamma/2) = \cot \gamma/2,$$
$$OA \cdot OC = \cot \alpha/2 \cot \gamma/2 = k^2$$

or

$$\tan \alpha/2 \tan \gamma/2 = 1/k^2. \tag{1}$$

Therefore, if two points on a sphere are inverse to each other with respect to a circle centered on a pole, then the product of the tangents of half their polar distances is a constant determined by the radius of the circle of inversion.

2. The Peaucellier linkage in the plane. In the plane, the Peaucellier linkage may be used to draw a straight line. The linkage contains an inversion linkage which locates the inverse of an arbitrary point with respect to a fixed circle. If a set of arbitrary points lies along a circle which passes through the center of the fixed circle, then the set of their inverse points lies along a straight line [**3**, **4**].

3. The Peaucellier linkage on the sphere. The inversion linkage of the Peaucellier linkage may be simulated on the sphere by four equal arcs of a spherical rhombic linkage of which two opposite vertices are joined by two equal spherical links to an exterior fixed point, as has been suggested by Duncan [**5**]. The following demonstration will show that the other pair of vertices of the rhombic linkage are inverse points on the sphere for all positions of the linkage. In Figure 2, $AB = BC = CD = DA$, $OB = OD$, and the fixed point is O. Then,

$$\left.\begin{aligned} OA \cdot OC &= (OE - AE)(OE + AE) = OE^2 - AE^2 \\ \frac{OA}{2} \cdot \frac{OC}{2} &= \left(\frac{OE - AE}{2}\right)\left(\frac{OE + AE}{2}\right) \end{aligned}\right\} \text{in the plane and on the sphere.}$$

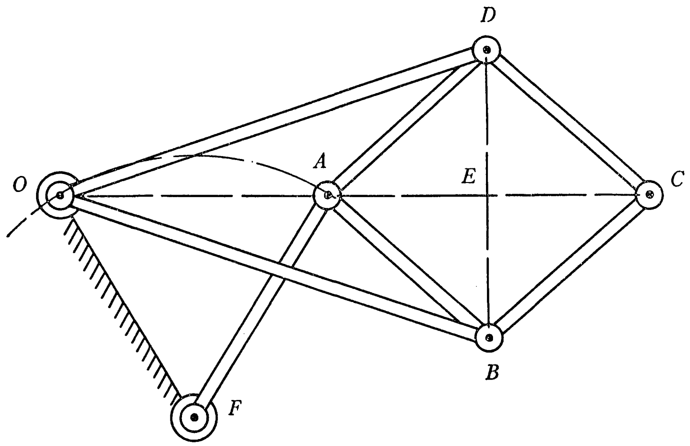

FIG. 2. Peaucellier linkage in the plane or on the sphere.

Therefore, on the sphere,

$$\cos (OA/2) = \cos [(OE/2 - (AE/2)] \quad \text{and}$$

$$\cos (OA/2) \cos (OC/2) = \cos [(OE/2) - (AE/2)] \cos [(OE/2) + (AE/2)]$$

$$= (\cos OE + \cos AE)/2.$$

From the spherical triangles ODE and ADE, we obtain

$$\cos OE = \cos OD/\cos DE \quad \text{and} \quad \cos AE = \cos AD/\cos DE.$$

Hence, $2 \cos (OA/2) \cos (OC/2) = \cos OD/\cos DE + \cos AD/\cos DE = (\cos OD + \cos AD)/\cos DE$. But, from triangle ADE, $\cos DE = \cos AD/\cos (AC/2)$. If $OA = \alpha$, and $OC = \gamma$, then $2 \cos \alpha/2 \cos \gamma/2 = (\cos OD + \cos AD) \cos (AC/2) /\cos AD$ and $(2 \cos \alpha/2 \cos \gamma/2)/\cos (AC/2) = (\cos OD + \cos AD)/\cos AD = k_1$ since OD and AD are fixed lengths of arc.

But $AC/2 = \alpha/2 - \gamma/2$. Therefore $(2 \cos \alpha/2 \cos \gamma/2)/(\cos \alpha/2 \cos \gamma/2 + \sin \alpha/2 \sin \gamma/2) = k_1$ and $2/(1 + \tan \alpha/2 \tan \gamma/2) = 1/k_1$. Hence

$$\tan \alpha/2 \tan \gamma/2 = 2k_1 - 1 = k_2 \tag{2}$$

where $k_2 = \tan (OD + AD)/2 \tan (OD - AD)/2$. Equation (2) has the same form as equation (1).

In the plane, the Peaucellier inversion linkage can be made into a straight line mechanism by fixing O, and constraining A to move in a circle through O. This is done by the addition of another link AF rotating about a fixed point F, where $AF = OF$. Then the point C describes a straight line. However, on the sphere, the point C describes a small circle through the opposite pole of O, or, in special cases, a great circle through the opposite pole.

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1. W. J. M'Clelland and T. Preston, A treatise on spherical trigonometry, part II (1896), chap. xii.
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3. H. Martyn Cundy and A. P. Rollett, Mathematical models, Clarendon Press, Oxford, 1952, 202–203.
4. A. B. Kempe, "How to draw a straight line," London, 1877, reprinted in "Squaring the circle and other monographs," Chelsea, New York (1953) 12–15.
5. W. J. Duncan, A kinematic property of the articulated quadrilateral, Quart. J. Mech. Appl. Math., 7, Pt. 2 (1954) 222–225.

BOOK REVIEWS

EDITED BY DMITRI THORO, San Jose State College

Materials intended for review should be sent to: Dmitri Thoro, Department of Mathematics, San Jose State College, San Jose, California 95114.

Limits, The Concept and its Role in Mathematics. By Norman Miller. Blaisdell, New York, 1964, 149 pp. $2.25 (paper).

Limits and Continuity. By William K. Smith. Macmillan, New York, 1964, 136 pp. $3.00 (paper).

Functions, Limits, and Continuity. By Paulo Ribenboim. Wiley, New York, 1964, 140 pp. $5.95.

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The first book, *Limits, The Concept and its Role in Mathematics*, by Norman Miller, is intended primarily for high school students and their teachers. The book begins with a brief but interesting historical introduction, and other comments on the history of the ideas considered are interspersed throughout the book. Certain properties of the real numbers are stated, and the notion of countability of sets is introduced, prior to the formal discussion of sets. Functions are defined in terms of ordered pairs. The chapters which follow deal with limits of sequences, limits of functions, infinite series, the derivative, and the definite integral. There are appendices discussing construction of the rational numbers from the integers (without formal consideration of equivalence classes), limit of a bounded monotone sequence, limits of Darboux sums, and limits of functions of more than one variable. Altogether, it seems an ambitious undertaking, considering the audience for whom it is intended.

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The student is led to the precise definition of the limit of a sequence in a gradual manner, and several examples are given illustrating the definition. The need for the precision of language is illustrated in the proof that the arithmetic means of the terms of a null sequence also form a null sequence. In a similar manner the limit of a function is defined after intuitive considerations. It seemed to me that a fuller explanation of the reason for requiring $x \neq a$ in defining $\lim_{x \to a} f(x) = L$ would be helpful to the student. The presentation of both of these limit concepts is not essentially different from that of most present day calculus books, but the comments given and the language used perhaps make this more easily understood by high school students.

The treatment of infinite series is fairly standard as far as it goes, including work through the ratio and comparison tests for convergence of series of positive terms. The derivative also is introduced in a standard way, and applications in maxima, minima, and calculation of asymptotes are given. The definite integral is defined in terms of upper and lower sums, and the fundamental theorem of calculus is presented. These last two chapters, on differentiation and integration, are the best in the book, in my opinion, giving more motivation for the ideas presented than is the case in previous chapters.

The basic idea of the author in presenting the limit concept and following this with some of its applications seems to have merit. However, the student might also wonder about the applications that are given. For example, nowhere is he given any hint as to why one would wish to study infinite series. The book deals with some of the most difficult ideas of calculus without including the wealth of applications which make the subject so fascinating. The desirability of such a course for high school students seems open to question. This of course is not a criticism of the book. For those wishing to give a course of this nature to high school students, this book should serve their purposes well. One thing is sure: the task of the teacher of calculus in college would be greatly reduced if his students had mastered the contents of this book. The book could also well be used for self study by high school teachers who are weak on these fundamental ideas.

The second little book in this trilogy, *Limits and Continuity*, by William K. Smith is intended as a supplement for the student of elementary or advanced calculus to aid in his understanding of the concepts of limits of functions and continuity of functions. It is one of the best books of this kind I have seen.

After an interesting introduction some of the language and symbolism of sets are introduced, and a function is defined as a mapping from one set into another. Work with inequalities and absolute value also is included, prior to the introduction of the main concepts.

The definition of limit of a function is arrived at through a series of five "attempts," each version more refined than the preceding one, and the *need* for each successive improvement is carefully pointed out. It is clear that Professor Smith has had a considerable amount of experience trying to explain this concept to beginning students. The final definition is phrased in terms of neighborhoods.

The next chapter deals with extensions of the basic limit definition to include various forms involving $\pm\infty$, limits of sequences, etc. Continuity is then introduced, and a chapter illustrating various ϵ-δ techniques, with numerous worked-out examples is presented.

The basic limit theorems and deeper properties of continuous functions are next presented. Use is made of the completeness property of the real numbers, and this is stated but not proved. Theorems on sequences and uniform continuity complete the book.

Many exercises are included both for drill and to extend the theory. I believe that the author has been successful in attaining the goal he set, and this book should prove to be a valuable supplement for calculus students at all levels.

The book by Paulo Ribenboim, entitled, *Functions, Limits, and Continuity*, is the most extensive, and also the most sophisticated of the three being reviewed here. In the author's words, the book " . . . is especially aimed at students of mathematics and physics (even gifted high-school students) who feel the need of understanding rather than calculating."

Following a brief introductory chapter on sets and correspondences, the rational numbers are constructed from the integers, which are taken as known. (The author shows in an exercise how to construct the integers, beginning from Peano's postulates.) The real numbers are then defined as equivalence classes of Cauchy sequences of rational numbers. Dedekind's construction of the reals, and also the axiomatic definition, are presented in an appendix. Many of the properties of the real numbers are to be derived in the exercises by the student.

Limits of sequences are dealt with next in a systematic manner, after proofs of the basic theorems on bounded sets of real numbers. The chapters which follow deal with the function concept, limits of functions, continuity, and uniform continuity. The Bolzano-Weierstrass and Heine-Borel theorems are proved and used extensively.

Professor Ribenboim has an incisive style of writing, which is beautiful in its simplicity. He has a remarkable facility for presenting difficult ideas in an easily understood manner. This book should be an excellent one for a freshman honors course, or for a course just preceding advanced calculus. It might also be suitable for certain summer institute courses for teachers.

On the first reading only one error was detected. On page 25 the statement is made that for α, β real numbers, n any nonzero integer, $\alpha^n=\beta^n$ implies $\alpha=\beta$.

LEONARD HOLDER, Gettysburg College

Polyominoes. By Solomon W. Golomb. Charles Scribner's Sons, New York, 1965. 182 pp. $5.95.

Recreational mathematics has an important part to play in education, firing the imagination and giving an attractive appearance to what may otherwise seem just drab theory. Too often, however, those who have written on recreational subjects have not treated any one aspect in depth, but have gone over the same extensive but familiar ground. In contrast, Professor Golomb has devoted a great deal of attention to polyominoes, with very rewarding results. It is a measure of the interest of his work that readers of this review can be assumed to know what polyominoes are. While much of the material has appeared elsewhere, in the sadly missed *Recreational Mathematics Magazine*, for example, it is most useful to have it collected together and expanded into book form. An almost ideal balance has been struck between the educational and recreational, between combinatorial geometry and entertaining puzzles. Few students could next presented. Use is made of the completeness property of the real numbers, and this is stated but not proved. Theorems on sequences and uniform continuity complete the book.

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The first chapter introduces polyominoes, with diagrams of those up to hexominoes, and gives some elementary results based mainly on chessboards and coloring arguments. The second chapter deals mainly with ten constructions using the 12 pentominoes, while the third gives details of patterns that exclude pentominoes from a chessboard: a variety of subtle and attractive methods of proof are used in these two chapters. Chapter IV explains methods of exhaustive search, and applies them to two difficult proofs of the nonexistence of pentomino arrangements. The next chapter, 'Some Theorems about Counting,' is an introduction to combinatorics and symmetry groups, with special reference to geometric arrangements and coloring, ending with Burnside's formula. There are over 50 carefully graded examples in this chapter, with answers at the end of the book. From the teaching point of view this is the heart of the book: it has been well prepared for by the earlier chapters and itself prepares the reader for the last two chapters. These deal with various extensions and generalisations—larger numbers of units, in more dimensions, and of different shapes. A theory of colored trees is introduced to represent polyominoes in three and more dimensions. There is an appendix where the problems of fitting polyominoes given throughout the book are collected together, with some others, including several that remain unsolved. There is a glossary and a bibliography, but no index.

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It is fitting to end this review, in the spirit of the book, with something for the reader to do. Prove that if three separate rectangles are formed simultaneously with some or all of the 12 pentominoes, then one of them is the single piece, 1×5. For what values of n are there such arrangements using just n pentominoes?

ALAN SUTCLIFFE, Congleton, Cheshire, England

The Mathematics of Great Amateurs. By Julian Lowell Coolidge. Dover, New York, 1963. viii+211 pp., paperbound $1.50.

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This volume is an unabridged republication of the 1949 Oxford University Press edition.

CHARLES W. TRIGG, San Diego, California

BRIEF MENTION

Arithmetic: A Modern Approach. By Bevan K. Youse. Prentice-Hall, Englewood Cliffs, N. J., 1963. xiv+160 pp. $4.95.

For in-service elementary-school teachers and college students majoring in elementary education.

Fundamentals of Modern Mathematics. By Jean M. Calloway. Addison-Wesley, Reading, Mass., 1964. x+213 pp. $6.75.

Written primarily for nonscience liberal arts students. No mathematical prerequisite. Ideas and understanding stressed. Chapters on calculus, probability, and mathematical systems.

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Lattices to Logic. By Roy Dubisch. Blaisdell, New York, 1964. vii+88 pp. $1.65 (paper).

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The Second All-Russian Olympiad in Mathematics. By Dmitri Thoro. J. Weston Walch, Box 1075, Portland, Maine, 1963. iv+36 pp. $1.00 (paper).

For the student or teacher who delights in simple but sophisticated problems. Extensive hints, supplementary problems, and detailed solutions.

College Algebra and Basic Set Theory, 2nd ed. By the Committee on College Algebra, J. C. Eaves, General Editor. Pitman, New York, 1963. xvi+459 pp. $7.50.

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A self-instructional program which could be completed within five hours.

PROBLEMS AND SOLUTIONS

EDITED BY ROBERT E. HORTON, Los Angeles City College

Readers of this department are invited to submit for solution problems believed to be new that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and exactly the size desired for reproduction.

Send all communications for this department to Robert E. Horton, Los Angeles City College, 855 North Vermont Avenue, Los Angeles, California 90029.

PROPOSALS

Erratum: In Problem 588 [May, 1965] the first operator should read $(D-1)^n x(D-1)$.

Trigonometry and the Transcendental Functions. By Tomlinson Fort. Macmillan, New York, 1963. viii+216 pp. $5.25.

Features a careful discussion of angles, numbers and measurement. A conventional treatment of the exponential, logarithmic and trigonometric functions is followed by an infinite power series approach.

Lattices to Logic. By Roy Dubisch. Blaisdell, New York, 1964. vii+88 pp. $1.65 (paper).

Lattice theory is introduced through a series of concrete examples from widely different fields. Includes exercises with completely worked-out answers.

The Second All-Russian Olympiad in Mathematics. By Dmitri Thoro. J. Weston Walch, Box 1075, Portland, Maine, 1963. iv+36 pp. $1.00 (paper).

For the student or teacher who delights in simple but sophisticated problems. Extensive hints, supplementary problems, and detailed solutions.

College Algebra and Basic Set Theory, 2nd ed. By the Committee on College Algebra, J. C. Eaves, General Editor. Pitman, New York, 1963. xvi+459 pp. $7.50.

A revision of the 1956 edition.

Basic Technical Mathematics with Calculus. By A. J. Washington. Addison-Wesley, Reading, Mass., 1964. xii+595 pp. $12.50.

Primarily for students taking technical programs at two-year colleges and institutes. Sufficient material for three or four semesters. Aimed at developing a student's feeling for mathematical methods.

Plane Trigonometry, Revised Edition. By F. A. Rickey and J. P. Cole. Holt, Rinehart and Winston, New York, 1964. ix+226 pp. $4.50.

A revision of the 1958 edition.

Computer-Oriented Mathematics. By L. D. Kovach. Holden-Day, San Francisco, 1964. vi+98 pp. $3.95 (cloth), $2.95 (paper).

Introduction to numerical methods including relaxation and Monte Carlo techniques.

Basic Slide Rule Operation: A Program for Self-Instruction. By W. S. Mittelstadt. McGraw-Hill, New York, 1964. vii+161 pp. $4.95 (cloth), $2.95 (paper).

A self-instructional program which could be completed within five hours.

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Erratum: In Problem 588 [May, 1965] the first operator should read $(D-1)^n x(D-1)$.

600. *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

If the area of a triangle ABC is S and the areas of the in- and ex-contact triangles are T, T_a, T_b, T_c, then show that

$$T_a + T_b + T_c - T = 2S \tag{1}$$

$$T_a^{-1} + T_b^{-1} + T_c^{-1} - T^{-1} = 0. \tag{2}$$

601. *Proposed by David Singmaster, University of California, Berkeley.*

Let ϕ be Euler's function. It is well known that $\phi(x) = 14$ has no solutions and that 14 is the smallest even number with this property. Show that there are infinitely many integers m such that the equation $\phi(x) = 2m$ has no solutions.

602. *Proposed by Bruce W. King, SUNY at Buffalo, New York.*

Show that

$$\sum_{i=0}^{n-2} \binom{n}{i} (-x)^{n-i}(x + \lambda)^i(n - i - 1)$$

gives all but the last two terms of the expansion of $(x+\lambda)^n$.

603. *Proposed by C. S. Venkataraman, Trichur, South India.*

The center of a circle is A, B a point outside it, and C a point on its circumference. BD and BE are tangents and the perpendicular at C to BC meets AD, AE in F, G respectively. If FM and GN are the perpendiculars to AB, prove:

(a) Triangles ABF and AGB are similar

(b) M and N are inverse points with respect to the circle.

604. *Proposed by Michael Gemignani, University of Notre Dame.*

Let R be a commutative ring with unity, and End R^+ be the ring of additive endomorphisms of R.

(a) Prove that End R^+ is a ring isomorphic to R if and only if it is commutative.

(b) Does the conclusion of (a) follow if we do not require R to have a unity?

605. *Proposed by Sam Newman, Atlantic City, New Jersey.*

A square island is surrounded by a square moat of water of width x feet. Explorers wishing to get to the island have available only two boards of length y feet ($y < x$) of negligible width. What is the largest width the moat can be for them to reach the island using these boards?

606. *Proposed by Gilbert Labelle, Université de Montreal, Canada.*

Evaluate

$$\int_0^1 \frac{1 - x^2}{1 + x^2} \cdot \frac{dx}{\sqrt{\{1 + x^4\}}}\,.$$

SOLUTIONS

LATE SOLUTIONS

572. *Frank Mauz, Western Michigan University.*

The Sultan's Heirs

579. [March, 1965] *Proposed by David L. Silverman, Beverly Hills, California.*

"If two of my children are selected at random, likely as not, they will be of the same sex," said the Sultan to the Caliph. "What are the chances that both will be girls?" asked the Caliph. "Equal to the chance that one child selected at random will be a boy," replied the Sultan. How many children did he have?

Solution by R. E. Nickels, Ferntree Gully Technical School, Victoria, Australia.

It is implied that the probability that two children selected at random will be of the same sex is $\frac{1}{2}$, therefore we have:

$$\left(\frac{x}{x+y}\right)\left(\frac{x-1}{x+y-1}\right)+\left(\frac{y}{x+y}\right)\left(\frac{y-1}{x+y-1}\right)=\frac{1}{2}\,. \tag{1}$$

Also, the probability that both will be girls is equal to the probability that one child selected at random will be a boy, therefore:

$$\left(\frac{y}{x+y}\right)\left(\frac{y-1}{x+y-1}\right)=\left(\frac{x}{x+y}\right). \tag{2}$$

From (1) and (2), we arrive at:

$$y=2(x-1) \tag{3}$$

which, when substituted in (2), gives:

$$x^2-7x+6=0. \tag{4}$$

Solving (3) and (4) gives the solutions: $x=1$, $y=0$; and $x=6$, $y=10$. The first solution is inadmissible because there must be more than one child, therefore, the Sultan had 16 children (6 boys and 10 girls).

Also solved by Merrill Barneby, University of North Dakota; Dermott A. Breault, Sylvania Applied Research Laboratory, Waltham, Massachusetts; David Chale, University of California, Berkeley, California; R. J. Cormier, Northern Illinois University; Richard K. Dawson, Lawrence, Massachusetts; Frank M. Eccles, Phillips Academy, Andover, Massachusetts; Michael Goldberg, Washington, D. C.; Carl Harris, Polytechnic Institute of Brooklyn; Roy H. Hines, Jr., Concord, Massachusetts; Robert F. Jackson, Toledo, Ohio; Richard A. Jacobson, South Dakota State University; Bruce W. King, Burnt Hills-Ballston Lake High School, New York; Sam Kravitz, East Cleveland, Ohio; E. L. Magnuson, HRB-Singer, Inc., State College, Pennsylvania; Brook Newcomb, Andover, Massachusetts; Charles J. Parry, Oswego, New York; Michael J. Pascual, Watervliet Arsenal, New York; Sidney Spital, California State Polytechnic College; Calvin C. Rice, Vestal, New York; Graham C. Thompson, Binghamton, New York; Lee Thorsen; C. W. Trigg, San Diego, California; A. M. Vaidya, Texas Technological College; K. L. Yocum, South Dakota State University; and the proposer.

A Diophantine Cubic

580. [March, 1965] *Proposed by Joseph Arkin, Spring Valley, New York.*

Is a solution in integers possible for the equation $(c-a-b)^3=24\,abc$, where a, b and c are not zero?

Solution by Stanley Rabinowitz, Far Rockaway, New York.

I shall make use of the identity

$$24\,abc = (a+b+c)^3 - (a-b+c)^3 - (-a+b+c)^3 + (c-a-b)^3.$$

Substituting this in the given equation,

$$(c-a-b)^3 = 24\,abc,$$

gives

$$(a+b+c)^3 = (a-b+c)^3 + (-a+b+c)^3.$$

But it is known that the equation $x^3+y^3=z^3$ has no integral solutions unless x, y, or z is zero which would imply that a, b, or c were zero.

Hence, the given equation has no nontrivial integer solutions.

Also solved by Graham C. Thompson, Binghamton, New York; and the proposer.

Complex Numbers

581. [March 1965] *Proposed by Joseph L. Teeters, Baker University, Kansas.*

If a complex number $a+bi$ is defined

I. to be positive when (i) $b>0$ or (ii) $b=0$ and $a>0$
II. to be zero when $b=0$ and $a=0$, and
III. to be negative when (i) $b<0$ or (ii) $b=0$ and $a<0$

and if $A<B$ (A, B being complex) means that $B-A$ is positive, then prove or disprove the following:

1. If A, B, C are complex numbers, and $A<B$, then $A+C<B+C$.
2. If A, B, C are complex numbers, and $A<B$, and C is positive, then $AC<BC$.

Solution by Frank Eccles, Phillips Academy, Andover, Massachusetts.

For Part 1, we have

$$\begin{aligned} A < B &\Rightarrow (B-A) \text{ is positive} \\ &\Rightarrow (B-A+0) \text{ is positive} \\ &\Rightarrow [(B-A)+(C-C)] \text{ is positive} \\ &\Rightarrow [(B+C)-(A+C)] \text{ is positive} \\ &\Rightarrow A+C<B+C. \end{aligned}$$

Part 2. This is disproved by a counter example. Let $A=0+2i$, $B=0+3i$, and $C=0+i$.

$$(B - A) = (0 + 3 - i) - (0 + 2i)$$
$$= 0 + i.$$

Since $0+i$ is positive, $B-A$ is positive and $A<B$.

Also solved by Roy Bowman, Western Michigan University; Martin J. Brown, St. Xavier High School, Cincinnati, Ohio; A. S. Galbraith, Baltimore, Maryland; Robert F. Jackson, Toledo, Ohio; Richard A. Jacobson, South Dakota State University; Erwin Just, Bronx Community College; Bruce W. King, Burnt Hills-Ballston Lake High School, New York; Brook Newcomb, Andover, Massachusetts; Stanley Rabinowitz, Far Rockaway, New York; L. B. Robinson, Baltimore, Maryland; Sidney Spital, California State Polytechnic College; A. M. Vaidya, Texas Technological College; Howard L. Walton, Yorktown High School, Arlington, Virginia; K. L. Yocum, South Dakota State University; and the proposer.

An Octahedron Section

582. [March, 1965] *Proposed by Charles W. Trigg, San Diego, California.*

A regular octahedron, edge e, is cut by a plane parallel to one of its faces. Find:

(a) the perimeter, and

(b) the area of the section.

Solution by Sidney Spital, California State Polytechnic College, Pomona, California.

Since opposite pairs of faces (equilateral triangles) of a regular octahedron are parallel, a plane parallel to one of its faces intersects six of the other faces. Let the intersection occur at a distance θe, $(0 \leqq \theta \leqq 1)$ from either one of the parallel faces, *measured along the edge of any one of the triangles intersected.* (Symmetry demands that the result be independent of which parallel faces or which intersected edge is chosen.) Examination of the curve of intersection shows it to be a hexagon, with three sides of length θe alternately neighboring three sides of length $(1-\theta)e$, and all internal angles equal to 120°. Simple calculations then show that for this hexagonal intersection:

$$\text{Perimeter} = 3e$$
$$\text{Area} = \frac{\sqrt{3}}{4} e^2[1 + 2\theta(1 - \theta)].$$

Also solved by Martin J. Brown, St. Xavier High School, Cincinnati, Ohio; R. J. Cormier, Northern Illinois University; Michael Goldberg, Washington, D. C.; Robert F. Jackson, Toledo, Ohio; Richard Laatsch, Miami University, Ohio; and the proposer. One incorrect solution was received.

An Infinite Sum

584. [March, 1965] *Proposed by J. Barry Love, Eastern Baptist College, Pennsylvania.*

Sum the series for $|x| > 1$,

$$\frac{1}{x+1} + \frac{2}{x^2+1} + \frac{4}{x^4+1} + \frac{8}{x^8+1} + \cdots.$$

I. *Solution by Mrs. A. C. Garstang, Boulder, Colorado.*

Put $y=x^{-1}$ so that $|y|<1$. Then the sum S becomes

$$\frac{S}{y}=\frac{1}{1+y}+\frac{2y}{1+y^2}+\frac{4y^3}{1+y^4}+\frac{8y^7}{1+y^8}+\cdots$$

Integrate term by term (the series is uniformly convergent):

$$\int^y \frac{S}{y}\,dy=\log(1+y)+\log(1+y^2)+\log(1+y^4)+\log(1+y^8)+\cdots+C.$$

Now

$$(1-y)(1+y)(1+y^2)(1+y^4)\cdots=1.$$

Hence

$$\int^y (S/y)dy=-\log(1-y)+C$$

and

$$S=y/(1-y)=1/(x-1).$$

II. *Solution by Eldon Hansen, Lockheed Missiles and Space Company, Palo Alto, California.*

The series

$$\sum_{k=0}^{\infty}\frac{2^k}{1+x^{2^k}}=\frac{1}{x-1}\qquad(|x|>1) \tag{1}$$

is well known. It is given as equations (91) and (301) of [**1**], equation 6.350 (3) of [**2**], and equation 1.121 (2) of [**3**]. It is also given on page 77 of [**3**].

Letting $x=1/y$, we see that equation (1) becomes

$$\sum_{k=0}^{\infty}\frac{2^k y^{2^k}}{1+y^{2^k}}=\frac{y}{1-y}\qquad(|y|<1).$$

This form is given as equation (88) or [**1**], equation 6.350 (2) of [**2**], and equation 1.121 (1) of [**3**]. It is also given as problem 3, page 123, of [**5**] and as problem 17, page 25, of [**6**].

Equation (1) is easily derived by noting that

$$\Delta\left(\frac{2^k}{1-x^{2^k}}\right)=\frac{2^k}{1+x^{2^k}}$$

where Δ is the forward difference operator. Hence,

$$\sum_{k=0}^{n-1}\frac{2^k}{1+x^{2^k}}=\frac{2^n}{1-x^{2^n}}-\frac{1}{1-x}.$$

Assuming $|x|>1$, we let $n\to\infty$ and obtain equation (1).

References

1. L. B. W. Jolley, Summation of series, 2nd ed., Dover, New York, 1961.
2. Edwin P. Adams and R. L. Hippisley, Smithsonian mathematical formulae and tables of elliptic functions, Smithsonian Inst., Wash. D. C., 1947.
3. I. M. Ryshik and I. S. Gradstein, Tables of series, products, and integrals, 2nd ed., Deutscher Verlag der Wissenschaften, Berlin, 1963.
4. J. M. Hyslop, Infinite series, Interscience, New York, 1959.
5. I. I. Hirshman, Jr., Infinite series, Holt, Rinehart, & Winston, New York 1962.
6. T. J. Bromwich, An introduction to the theory of infinite series, Macmillan, New York, 1947.

Also solved by Charles L. Belna, University of Dayton, Ohio; L. Carlitz, Duke University; Allan Chuck, San Francisco, California; R. J. Cormier, Northern Illinois University; Kenneth Dieter, Medford, Massachusetts; A. S. Galbraith, Baltimore, Maryland; Michael Goldberg, Washington, D. C.; Robert F. Jackson, Toledo, Ohio; Erwin Just, Bronx Community College; Stanley Rabinowitz, Far Rockaway, New York; Henry J. Ricardo, Yeshiva University, New York; Morris E. Rill, University of Oklahoma; Sidney Spital, California State Polytechnic College; Lee Thoresen; A. M. Vaidya, Texas Technological College; John Waddington, Levack, Ontario, Canada; Raymond E. Whitney, Lock Haven State College, Pennsylvania; K. L. Yocum, South Dakota State University; and the proposer. A number of other references to the problem were given.

A Trinomial Distribution

585. [March, 1965] *Proposed by J. M. Howell, Los Angeles City College.*

A population consists of three types of objects. Let P, Q and R represent the probability of drawing one of each type on a single draw, $P+Q+R=1$. A sample of size n is drawn with replacement, and p, q, r represent the fractions of these three types of objects found in the sample. The mean and variance of $p+r$ would be $P+R$ and $(P+R)Q/n$, since this would be a binomial distribution. What are the mean and variance of $p-r$?

I. *Solution by Sidney Spital and Cameron Bogue (jointly), California State Polytechnic College, Pomona, California.*

Since the joint distribution of the random variables $\{np, nq, nr\}$ is trinomial of sample size n with probability parameters $\{P, Q, R\}$, it is known that

$$\text{mean}\,\{p, q, r\} = \{P, Q, R\}$$

$$\text{var}\,\{p, q, r\} = \left\{\frac{P(1-P)}{n}, \frac{Q(1-Q)}{n}, \frac{R(1-R)}{n}\right\}$$

$$\text{cov}\,\{pq, qr, rp\} = \left\{-\frac{PQ}{n}, -\frac{QR}{n}, -\frac{RP}{n}\right\}.$$

The desired results then follow directly:

$$\text{Mean}\,(p-r) = \text{mean}\,(p) - \text{mean}\,(r) = P - R$$

$$\text{var}\,(p-r) = \text{var}\,(p) + \text{var}\,(r) - 2\,\text{cov}\,(pr)$$

$$= \frac{1}{n}\,(P(1-P) + R(1-R) + 2RP)$$

$$= \frac{1}{n}(P + R - (P - R)^2).$$

II. *Solution by James C. Hickman, University of Iowa.*

Let X be the number of items selected from the class associated with probability P, and let Y be the number of items selected from the class associated with probability R. Then the moment generating function for $p-r$ is

$$E(\exp(t(X - Y)/n)) = (P\exp(t/n) + Q + R\exp(-t/n))^n.$$

Denoting this moment generating function by $M(t)$, we have that

$$E(p - r) = M'(t)\big|_{t=0} = P - R, \text{ and}$$

$$\operatorname{Var}(p - r) = M''(t)\big|_{t=0} - (M'(t)\big|_{t=0})^2$$

$$= \frac{(P + R)}{n} - \frac{(P - R)^2}{n}.$$

Also solved by Roy H. Hines, Concord, Massachusetts; and the proposer.

The proposer pointed out that $\sigma_{p-r} = K\sigma_{p+r}$ where

$$K^2 = 1 + \frac{4PR}{(P + R)Q}.$$

Thus σ_{p-r} can be expressed in terms of the binomial standard deviation and tables of that statistic may be used.

Comment on Problem 567

567. [November, 1964; and May, 1965] *Proposed by L. Carlitz, Duke University.*

Points A_1, A_2 are marked on the side BC of the triangle ABC so that $BA_1 = A_1A_2 = A_2C$, similarly B_1, B_2 on CA and C_1, C_2 on AB. Let A' be the point of intersection of BB_1 and CC_2, B' of CC_1 and AA_2, C' of AA_1 and BB_2. How is the triangle $A'B'C'$ related to ABC?

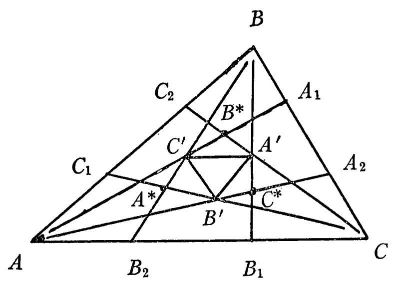

FIG. 1.

Comment by Ronald R. Schryer, University of California, Riverside.

As a means of solving the problem proposed by Carlitz, I have applied a method that was suggested by a talk, "Web of Mathematics," given by R. A.

Rosenbaum, Wesleyan University, at the Long Beach Conference of the California Mathematics Council, December 12, 1964. At that time, Professor Rosenbaum was considering a method of establishing the concurrence of the medians of a triangle by assigning equal weights to the vertices of the triangle and locating the centroid.

Before attacking the problem directly, it will be convenient to consider a rather general relation that can be established between the line segment $V_3' V_1'$ and the line segment $V_1 V_3$ in Figure 2.

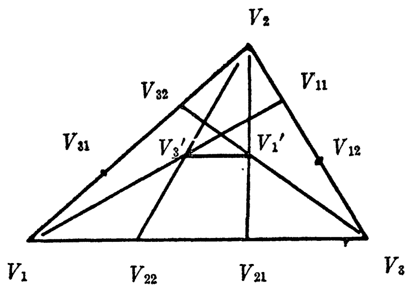

FIG. 2.

For the triangle of Figure 2,

$$V_2V_{11} = V_{11}V_{12} = V_{12}V_3,$$
$$V_3V_{21} = V_{21}V_{22} = V_{22}V_1,$$
$$V_1V_{31} = V_{31}V_{32} = V_{32}V_2.$$

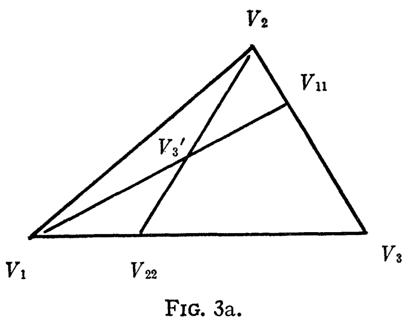

FIG. 3a.

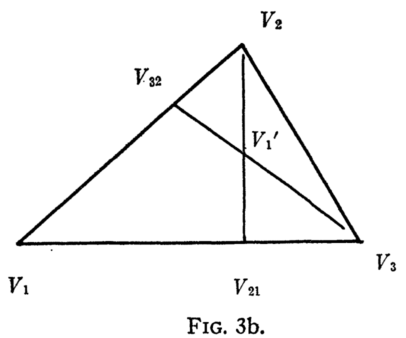

FIG. 3b.

If we consider the system shown in Figure 3a with weights of 2 at V_1 and V_2 and a weight of 1 at V_3, we find that the centroid of the system is V_3'. From a basic principle of mechanics, we realize that the system $V_1 V_3$ can be considered as if a weight of 3 is assigned to V_{22}; then as V_3' is the centroid of the system, we clearly have

$$\frac{V_2V_3'}{V_3'V_{22}} = \frac{3}{2}$$

or what is more important to us

$$\frac{V_2V_3'}{V_2V_{22}} = \frac{3}{5}.$$

Now by considering the system in Figure 3b and assigning weights of 2 to the vertices V_2 and V_3 with a weight of 1 assigned to the vertex V_1, we find that the centroid of this system is the point V_1'. Again we recognize that the V_1V_3 system may be considered as having its weight (3) concentrated at V_{21}. As V_1' is the centroid of this system, it is again clear that

$$\frac{V_2V_1'}{V_1'V_{21}} = \frac{3}{2}$$

or

$$\frac{V_2V_1'}{V_2V_{21}} = \frac{3}{5}.$$

Returning now to the triangle of Figure 2, we see that triangle $V_2V_1'V_3'$ is similar to the triangle $V_2V_{21}V_{22}$ with a constant of proportionality of 3/5. It is then quite apparent that

$$\frac{V_3'V_1'}{V_{22}V_{21}} = \frac{3}{5}$$

and since

$$\frac{V_{22}V_{21}}{V_1V_3} = \frac{1}{3}$$

the desired relationship that we were seeking is

$$\frac{V_3'V_1'}{V_1V_3} = \frac{1}{5}.$$

By application of the relationship just established, with each of the vertices ABC of the original triangle (Fig. 1) playing the role of V_2 we have that

$$\frac{C'B'}{CB} = \frac{C'A'}{CA} = \frac{A'B'}{AB} = \frac{1}{5}.$$

Triangle $A'B'C'$ is then seen to be similar to triangle ABC with a constant of proportionality of 1/5 and the ratio of the areas is clearly $(1/5)^2$ or 1/25.

A rather interesting observation regarding this problem comes from considering the triangle $A^*B^*C^*$ where A^* is the intersection of BB_2 and CC_1, B^* is the intersection of AA_1CC_2, and C^* is the intersection of AA_2 and BB_1. This triangle has an area 1/16 the area of the original triangle ABC, indicating that the subscripts of Figure 1 are not commutative. This fact may contribute to the lack of a neat solution to the general class of problems.

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q369. Find

$$I_n = D^n \left\{ \arc \tan \frac{2x^3}{1+3x^2} \right\}_{x=0}.$$

[*Submitted by M. S. Klamkin*]

Q370. Evaluate

$$\int_0^1 (\sqrt{\{2-x^2\}} - \sqrt{\{2x-x^2\}})dx$$

taking the positive values of both square roots.
[*Submitted by Alan Sutcliffe*]

Q371. In what bases are 35 and 58 relatively prime?
[*Submitted by David L. Silverman*]

Q372. Given a set of n lines $y_i = m_i x + b_i$, find the abscissa x that minimizes the sum of the lengths of the ordinates $\sum |y_i|$. What is this sum? Equivalent to "minimize the norm of the matrix $B + \lambda M$."
[*Submitted by Charles Maley and Gino Coviello*]

Q373. Show that e^x is a transcendental function.
[*Submitted by M. S. Klamkin*]

(Answers on page 302)

ACKNOWLEDGEMENT

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Shreeram Abhyankav, H. L. Alder, C. B. Allendoerfer, R. L. Blair, S. J. Bryant, Leonard Carlitz, W. B. Caton, P. A. Clement, Edward Davis, M. P. Drazin, Howard Eves, Bernard Friedman, Mason Henderson, Arlo Hendrickson, David Herzig, Jay Hook, Howard Hughes, J. A. H. Hunter, J. G. Kemeny, J. M. Kingston, V. L. Klee, Kwangil Koh, Anton Kraft, M. H. McAndrew, Anthony Patricelli, E. C. Posner, W. J. Purcell, R. G. Thompson, W. T. Tutte, G. N. Wollan, W. B. Woolf, and H. S. Zuckerman.

ANSWERS

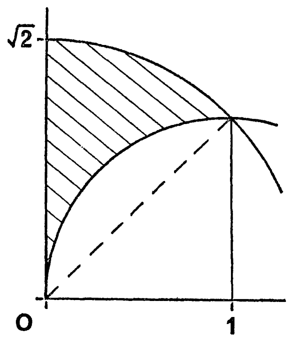
A369. Since

$$\text{Arc tan}\,\frac{2x^3}{1+3x^2} = 2 \text{ arc tan } x - \text{arc tan } 2x$$

we have $I_{2n}=0$ and

$$I_{2n+1} = \frac{(-1)^n}{2n-1}\,(2^{2n-1}-2).$$

A370. The value of the integral is 1/2. From the figure, the area under the lower curve is $\pi/4$. That under the higher curve, in the two parts divided by the dotted line, is $1/2\cdot 2\cdot\pi/4+1/2$. The difference is 1/2.



A371. In all bases since $(35, 58)=(35, 23)=(12, 23)=(12, 11)=(1, 11)=1$.

A372. $d/dx|y_i| = \pm m_i$ and

$$d/dx \sum |y_i| = \sum (\pm m_i)$$

with signs according as y_i is + or −, is a step function with $n+1$ treads, the risers being at the nx-intercepts. Thus x lies at one of those intercepts, specifically at $x_j=-b_j/m_j$, where the riser ascends through the x-axis. The sum is then

$$(1/m_j)\sum_i \begin{vmatrix} b_i & m_i \\ b_j & m_j \end{vmatrix}.$$

A373. Assume that e^x is algebraic, then

$$a_0(x)e^{nx} + a_1(x)e^{(n-1)x} + \cdots + a_n(x) = 0,$$

where $a_r(x)$ are polynomials. Consequently,

$$-a_0(x)e^{x/2} = \frac{a_1(x)e^{(n-1)x} + \cdots + a_n(x)}{e^{(n-1/2)x}}.$$

Letting $x\to 0$, we obtain a contradiction, whence e^x is transcendental.

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